CPSC 121,
2007 Summer
Quiz 4

Name: ____________________________ Student ID: ____________________

Signature: _________________________

- You have **20 minutes** to write the **2 questions** on this examination.
- A total of 20 marks are available. The marks for each question are shown in square brackets to the left of the question number. **You may want to complete what you consider to be the easiest questions first!**
- Justify all of your answers.
- No notes or electronic equipment are allowed.
- Keep your answers short. If you run out of space for a question, you have written too much.
- Use the attached blank page for your rough work.
- Good luck!

Question	Marks
1	
2	
Total	

UNIVERSITY REGULATIONS:

- Each candidate should be prepared to produce, upon request, his/her university-issued ID.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour or to leave during the first half hour of the examination.
- CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
    1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
    2. Speaking or communicating with other candidates.
    3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

[10] 1. Use mathematical induction to prove that for every n ≥ 1,

$$\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^n}=1-\left(\frac{1}{3}\right)^n$$

Some marks will be given for the structure of the proof and the remainder of the marks will be for the details.

Let P(n) = $\sum_1^n \frac{2}{3^n} = 1-\left(\frac{1}{3}\right)^n$

Base case: n =1

When n = 1, $\sum_1^n \frac{2}{3^n} = \frac{2}{3}$ and $1-\left(\frac{1}{3}\right)^n = 1-\frac{1}{3}=\frac{2}{3}$

So our base case holds

Inductive step: We must prove that ∀n ≥ 1, P(n) → P(n+1)

Inductive Hypothesis: We assume that P(n) is true, or that $\sum_1^n \frac{2}{3^n} = 1-\left(\frac{1}{3}\right)^n$

We must now show that P(n+1) must be true.

$\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^n}+\frac{2}{3^{n+1}} = 1-\left(\frac{1}{3}\right)^n+\frac{2}{3^{n+1}}$ (by the inductive hypothesis)

$= 1-\frac{1}{3^n}+\frac{2}{3^{n+1}}$ (rules of exponents)

$= 1-\frac{1}{3^n}\left(\frac{3}{3}\right)+\frac{2}{3^{n+1}}$ (multiply by $\frac{3}{3}$)

$= 1-\frac{3}{3^{n+1}}+\frac{2}{3^{n+1}}$

$= 1-\frac{1}{3^{n+1}}$ (subtract the fractions)

Since, P(1) ∧ ∀n ≥ 1, P(n) → P(n+1), by the principle of mathematical induction, ∀n ≥ 1, P(n)

[10] 2. Consider the sequence defined as follows:

$x_0 = 1$
$x_1 = 2$
$x_2 = 3$
$x_n = x_{n-3} + x_{n-2} + x_{n-1}$ for all $n \geq 3$

So, the sequence begins like this: 1, 2, 3, 6, 11, 20, ….

Using strong mathematical induction, prove that $x_n \leq 2^n$ for all $n \geq 0$.

(Hint: you may use the fact that $2^0 + 2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$)

Let $P(n) = x_n \leq 2^n$

Base cases: n = 0, 1 and 2

when n = 0, $x_0 = 1$ and $2^0 = 1$ so $x_0 \leq 2^0$
when n = 1, $x_1 = 2$ and $2^1 = 2$ so $x_1 \leq 2^1$
when n = 2, $x_2 = 3$ and $2^2 = 4$ so $x_2 \leq 2^2$

Inductive Step: Prove that $\forall n \geq 2$, $P(0) \wedge P(1) \wedge \ldots \wedge P(n) \rightarrow P(n+1)$
Inductive Hypothesis: Assume that $P(0) \wedge P(1) \wedge \ldots \wedge P(n)$ is true.
We must now show that P(n+1) must be true.

$x_{n+1} = x_{(n+1)-3} + x_{(n+1)-2} + x_{(n+1)-1}$

$x_{n+1} = x_{n-2} + x_{n-1} + x_n$

$x_{n+1} \leq 2^{n-2} + 2^{n-1} + 2^n$ (by IH)

$x_{n+1} \leq 2^0 + 2^1 + \ldots + 2^{n-3} + 2^{n-2} + 2^n = 2^{n+1} - 1$

$x_{n+1} \leq 2^{n+1}$

Therefore, since $[P(0) \wedge P(1) \wedge P(2)] \wedge \forall n \geq 2, P(0) \wedge P(1) \wedge \ldots \wedge P(n) \rightarrow P(n+1)$, by the principle of mathematical induction, $\forall n \geq 0$, P(n).