CPSC 121,
2005/6 Winter Term 2, Section 203
Quiz 2

Name: SAMPLE SOLUTION_____________ Student ID: ____________________________

Signature: ______________________________

- You have **30 minutes** to write the **3 questions** on this examination.
- A total of 40 marks are available. **You may want to complete what you consider to be the easiest questions first!**
- Justify all of your answers.
- No notes or electronic equipment are allowed.
- Keep your answers short. If you run out of space for a question, you have written too much.
- The number in square brackets to the left of the question number indicates the number of marks allocated to that question.
- Use the attached blank page for your rough work.
- Good luck!

Question	Marks
1	
2	
3	
Total	

UNIVERSITY REGULATIONS:

- Each candidate should be prepared to produce, upon request, his/her university-issued ID.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour or to leave during the first half hour of the examination.
- CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
  2. Speaking or communicating with other candidates.
  3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

[15] 1. Jubjubs have a height and a size. Jubjubs are formed in three ways:

(1) A new Jubjub of height 1 and size 1 appears every afternoon.

(2) Two Jubjubs of different heights can form a new Jubjub. The new Jubjub's size is the sum of the two parents' sizes and its height is the height of the taller parent.
NOTE: one of the two parents must be the same height as the child!

(3) Two Jubjubs of the same height can form a new, taller Jubjub. The new Jubjub's size is the sum of the two parents' sizes and its height is one larger than its parents' height.
NOTE: one of the two parents must be ≤ half the size of the child!

For example:

- Using (2), a Jubjub of height 5 and size 41 and a Jubjub of height 3 and size 9 can form a Jubjub of height 5 and size 50.
- Using (3), a Jubjub of height 3 and size 20 and a Jubjub of height 3 and size 8 can form a Jubjub of height 4 and size 28.

Using induction on the size of a Jubjub, prove the following for every Jubjub, where **h** is the Jubjub's height and **s** is the Jubjub's size:

$$h \leq (\log_2 s) + 1$$

Help with logs: $\log_2 1 = 0$. $(\log_2 n) + 1 = \log_2 (2n)$.
Hint: proceed in two cases for your inductive step based on how the Jubjub was formed!

BASE CASE: the only way to create a Jubjub of size 1 (or height 1) is the first method above, since any mating produces a larger Jubjub of at least the same height (and any mating using the second method above produces a taller Jubjub).

So, in the base case of $s_J = 1$, $h_J = 1$. And: $h_J = 1 \leq 1 = 0 + 1 = (\log_2 s_J) + 1$

INDUCTIVE STEP:

Assume that for a Jubjub of any size $1 \leq s_{J'} < n$, $h_{J'} \leq (\log_2 s_J) + 1$.

To prove: for a Jubjub of size n, $h_J \leq (\log_2 s_J) + 1$.

We will prove this by cases. Any Jubjub of size n must have been produced by either method 2 or 3 above (since method 1 only produces Jubjubs of size 1.) Therefore, case 1 will consider method 2, and case 2 will consider method 3.

Case 1: the Jubjub of size n had two parents of differing heights.
Call the height of the taller parent h_T and the size of the taller parent s_T. Similarly, the shorter parent has h_S and s_S.
Since the shorter parent is at least size 1, $s_J = s_T + s_S > s_T$.
(Note that $\log_2 s_1 \geq \log_2 s_2$ iff $s_1 \geq s_2$.)
By method 2 above, $h_J = h_T$.

Then, $h_J = h_T \leq (\log_2 s_T) + 1 < (\log_2 s_J) + 1$. QED for case 1.

Case 2: the Jubjub of size n had two parents with the same height.
Call the size of the smaller parent s_S and its height h_S. (Choose either if both are the same size.)
By method 3 above, $h_J = h_S + 1$.
By the note on method 3, $s_S \leq s_J / 2$.
(Proof: either the parents are the same size, in which case their size is exactly the child's size over 2. Otherwise, they are different sizes, in which case doubling the smaller size is not as big as adding the two sizes, i.e., not as big as the child's size.)

Now: $h_J = h_S + 1 \leq (\log_2 s_S) + 1 + 1 \leq (\log_2 (s_J / 2)) + 1 + 1 = (\log_2 (2 * s_J / 2)) + 1 = (\log_2 s_J) + 1$.
QED for case 2.

Thus, the claim is proven for all sizes of Jubjub.

[10] 2. Prove the following for arbitrary sets A, B, and C: $(A \cap B \subseteq C) \rightarrow B \subseteq \overline{A} \cup C$. Your proof may be in a combination of English and logic but make sure that it's clear what the steps of your proof are and how each step follows from the previous steps.

There are many possible proofs. My favourite is to convert to logical form, use antecedent assumption, and then remove conditionals and move negations inward in an effort to convert my assumption into my conclusion. Note that when actually working the problem, I go forward from the antecedent and also (carefully) work backward from the conclusion; however, a good final form of a proof should proceed in a straight line.

$(A \cap B \subseteq C) \rightarrow B \subseteq \overline{A} \cup C \equiv$

$\forall x \in U, (x \in A \wedge x \in B \rightarrow x \in C) \rightarrow (x \in B \rightarrow x \notin A \vee x \in C)$.

So, we will prove the logical statement.

Without loss of generality, pick an arbitrary $x \in U$.
We will then prove this by antecedent assumption.
Assume $x \in A \wedge x \in B \rightarrow x \in C$

$\neg(x \in A \wedge x \in B) \vee x \in C$	(by def'n of conditional)
$x \notin A \vee x \notin B \vee x \in C$	(by De Morgan's)
$x \notin B \vee (x \notin A \vee x \in C)$	(by commutative/associative laws)
$x \in B \rightarrow x \notin A \vee x \in C$	(by def'n of conditional)

QED.

[15] 3. Consider the set A = {hello, 3, ∅}, its power set P(A), and its Cartesian product with itself A × A. For each of the following functions, give an example value from the domain and its corresponding image. Then, say whether the function is (**Y**) or is not (**N**) injective, surjective, or bijective. The first row is filled in as an example.

There are MANY possible answers to some of these. In particular, ALL of the pre-image/image questions have multiple valid answers. (Remember that is ∅ equivalent to {}.)

The function `f`	Pre-image	Image	Injective? (**Y** or **N**)	Surjective? (**Y** or **N**)	Bijective? (**Y** or **N**)
`f : A → A` maps values onto themselves: `f(x) = x`.	`hello`	`hello`	Y	Y	Y
`f : A → (A ∪ P(A))` maps values onto themselves: `f(x) = x`.	hello	hello	Y	N	N
`f : A × A → A` picks out the first element from a 2-tuple	(3, {})	3	N	Y	N
`f : A → P(A)` maps elements into sets: `f(x) = {x}`.	3	{3}	Y	N	N
`f : P(A) → P(A)` "flips" sets: `f(B) = A - B`	{3,hello, {}}	{}	Y	Y	Y
`f` takes 2-tuples of elements from `A` and maps them into subsets of `A`. (**You** pick the particular function.)	(3,hello)	{3}	N	Y/N	N

Note that {} is a reasonable image in the fourth row, but NOT because it is a member of A! {} only works because it is the whole set A "flipped": A – A. Another pre-image/image pair would be {3} and {hello, {}}.

Note that there is NO function mapping `A × A` to `P(A)` that is injective (or bijective) because there are too many elements of `A × A` for each to get its own element of `P(A)`. However, the function can (potentially) be surjective.

Note that every function that is both injective and surjective is also bijective, and vice versa.