Midterm 2 Review

March 19th, 2012

1 Topics

i. Predicate Logic

ii. Formal Proofs & Techniques

iii. Deterministic Finite State Automata

iv. Flipflops, Circuit Memory

v. Induction

2 Examples

I Use the following to answer each question in two equivalent different ways, one existential, one universal:

- $\text{Dog}(x) : x$ is a dog
- $\text{Green}(x) : x$ is green-coloured
- $\text{LargerThan}(x, y) : x$ is larger than y
- $\text{SameSize}(x, y) : x$ is the same size as y
- A, the set of all animals

(a) There is an animal that is green.

$\exists x \in A, \text{Green}(x)$
$\sim\forall x \in A, \sim\text{Green}(x)$

(b) No dogs are green.

$\forall x \in A, \text{Dog}(x) \rightarrow \sim\text{Green}(x)$
$\sim\exists x \in A, \text{Dog}(x) \wedge \text{Green}(x)$

(c) There is a non-dog that is larger than a dog.

$\exists x \in A, \sim\text{Dog}(x) \wedge (\exists y \in A, \text{Dog}(y) \wedge \text{LargerThan}(x, y))$
$\sim\forall x \in A, \sim\text{Dog}(x) \rightarrow (\forall y \in A, \text{Dog}(y) \rightarrow \sim\text{LargerThan}(x, y))$

II Convert these proofs to predicate statements and outline how you would approach the proof:

(a) Is it true that if a and b are real numbers, $a \neq 0$, then there is a unique real number r s.t. $ar + b = 0$

$\forall a, b \in \mathbb{R}, a \neq 0 \rightarrow \exists r, s \in \mathbb{R}, r \neq s \wedge ar + b = 0$

Direct proof. Assume true, and use the second fact to move into a fraction. Then try finding values of r and try to ensure it is unique for another number (e.g. s) Turns out it doesn't work but...

(b) Prove that if m is a perfect square, $m + 2$ is not a perfect square

Let PSquare(x) : x is a perfect square.
$\forall m \in \mathbb{Z}$ PSquare(m) $\rightarrow$ PSquare($m + 2$)

Attempt direct proof. i.e.
Assume m is a perfect square, $m + 2$, e.g. $m = a^2$ for some $a \in \mathbb{Z}$
Work with $(a + 1)$ to show that we have to be at least $m + 3$ to work.

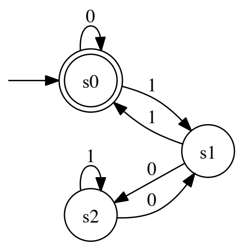
(c) Prove that if x^3 is irrational, x is irrational.

Let Irrational(x) : x is an irrational number.
$\forall x \in \mathbb{Z}$ Irrational(x^3) $\rightarrow$ Irrational(x)

Use contrapositive.
Assume x is not irrational, thus x is rational.
This means $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0$.
Get the form x^3 = some fraction.
Thus we can conclude by contrapositive that if x^3 is irrational, so must x be.

III Describe the languages the following DFA's accept:

(a)

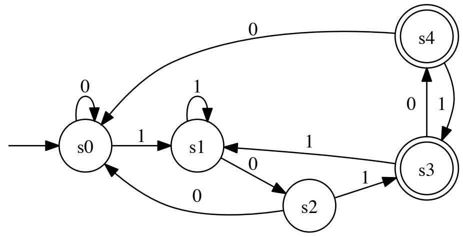


Binary numbers divisible by 3. A binary number is divisible by three if we take the even positioned 1's and subtract the odd positioned 1's from it. E.g. $100111_2(39_{10})$ has the 1st and 5th bits as 1's and the 4th and 6th bits also as ones so we get 2 - 2 = 0. As a formula this is:

Num_{odd} - Num_{even} mod(3) = 0.

Note: Writing "Any number with a pair of 1's OR Any number with a pair of 0's surrounded by two ones, with any amount of 1's in between the pair of 0's" is also correct, but not as preferable.

(b)



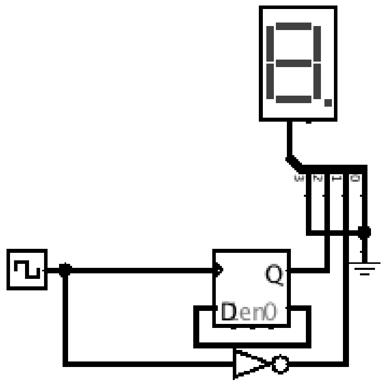
Numbers that end in 101 or 1010.

IV Consider the following Circuit questions:

(a) How does one get an 'initial' value for a flip-flop? How do we prevent using this value each clock-tick?

Feed in the clock value with an XOR gate or an OR gate for example. Basically if we feed in Q to D, initially Q is undefined, so we need a starting value and we can use the clock to 'seed' this. Of course Logisim will assume there is an initial value, hence why we can do a loop from ∼Q to D, as seen in a frequency divider.

(b) Design a circuit that counts even numbers up to 6 (i.e. 2, 4, 6, 0)



V Induction Questions:

(a) Prove $P(n) \leftrightarrow 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

Base case: $n = 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$
$$1 = 2^{(0+1)-1}$$
$$1 = 2 - 1$$
$$1 = 0$$

Inductive hypothesis: Assume this holds for $n = k$

We have

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

and we will now show this is true for $n = k + 1$

Scratch work:

$$\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{(k+1)+1} - 1$$
$$\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+2} - 1$$

Inductive Step: Starting from $n = k$

$$\begin{aligned}
\text{Right side of equation} &= 2^{k+1} - 1 \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+2} - 1
\end{aligned}$$

And thus our right hand side = left hand side.

Thus we have shown this statement applies for $n = 1$ and proves it for $n = k \rightarrow n = k + 1$.
It then follows that this holds for $n = 1, 2, 3, \ldots$

(b) $x, y, n \in \mathbb{N}$
If $x < y$, then $x^n < y^n$

Base case: $n = 1$
If $x < y$ then $x^1 < y^1$
This statement is correct, thus we move to the inductive hypothesis.

Inductive hypothesis: Assume this statement now holds for $n = k$
If $x < y$ then $x^k < y^k$

Scratch Work:
So obviously $x^{k+1} < y^{k+1}$ breaks into $x \cdot x^{k+1} < y \cdot y^{k+1}$
How do we get here?

Now we must show some supporting information:
Since $0 < a < b$ and $0 < x < y$, we can state $ax < bx$ and thus $bx < by$
By knowing this, we can state:
If $x^k < y^k$ then, by above, $(x^k)x < (y^k)y$
This is $x^{k+1} < y^{k+1}$
Since $x < y$ this statement must hold $\forall k$.

Thus we can state this holds $\forall n$ as long as $x, y, n \in \mathbb{N}$