CPSC 121 :: 2011S :: Midterm 2 :: 2011.07.18

NAME: SOLUTION
SIGNATURE:
STUDENT NUMBER:

- There are 9 pages in total.
- You have 60 minutes.
- A total of 50 (+1) marks are available.
- As a suggestion, allocate ≈1 minute per mark.
- You may want to complete what you consider to be the easiest questions first!
- No notes or electronic devices are permitted.
- Use the backs of pages if you require additional space, and clearly identify when you have done so.

UNIVERSITY REGULATIONS:

- Each candidate should be prepared to produce, upon request, his/her university-issued ID.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour or to leave during the first half hour of the examination.
- CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
  2. Speaking or communicating with other candidates.
  3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	TOTAL

Question 1. [10 marks]
average: **8.2** (median: 8.25, mode: 10)

Given:

- A is the set of all animals in the world
- $H(x) \Leftrightarrow x$ is a human (x is an animal, and humans are animals)
- $P(x) \Leftrightarrow x$ is a pig
- $D(x) \Leftrightarrow x$ is a dog
- $E(x, y) \Leftrightarrow x$ eats y

Rewrite the following statements formally without defining any new domains or predicates:

part a. [2 marks]
Bob is a human.

SOLUTION
H(Bob)

part b. [2 marks]
There are no humans in our world.

SOLUTION
$\forall x \in A, \sim H(x)$

part c. [2 marks]
The only animals in our world are dogs.

SOLUTION
$\forall x \in A, D(x)$

part d. [2 marks]
Bob is the only human in our world that eats pigs.

SOLUTION
$\forall x \in A, \forall y \in A, (H(x) \wedge P(y) \wedge E(x, y)) \rightarrow (x = \text{Bob})$

part e. [2 marks]
It's a dog eat dog world. (All dogs eat all other dogs).

SOLUTION
$\forall x \in A, \forall y \in A, (D(x) \wedge D(y) \wedge (x \neq y)) \rightarrow E(x, y)$

Question 2. [10 marks]
average: **5.5** (median: 4.5, mode: 3)
Re-write the following statement formally, and then prove or disprove the statement:

For any positive rational numbers x and z, where $x < z$, there exists a positive rational number y such that $x < y < z$.

SOLUTION

$$\forall x \in \mathbb{Q}^+, \forall y \in \mathbb{Q}^+, (x < z) \rightarrow (\exists y \in \mathbb{Q}^+, (x < y < z))$$

Direct Proof (Generalization)

WLOG, Let $x = a/b$ and $z = c/d$ be arbitrary rational numbers, where a, b, c, d are positive integers.

$\therefore\ x = a(2d)/b(2d), z = c(2b)/d(2b)$.

Choose $y = (x + y)/2 = (ad + bc)/2bd$ (right in-between them).

Since a, b, c, d are all positive integers, $(ad + bc)$ is an integer and $2bd$ is an integer,
$\therefore$ y is rational

Need to show $(x < z) \rightarrow (x < y < z)$
which is the same as $(2ad < 2bc) \rightarrow (2ad < ad + bc < 2bc)$

Step 1: show $(2ad < ad + bc)$:
if $(x < z)$
$\therefore\ 2ad < 2bc$
$\therefore\ ad < bc$
$\therefore\ ad + ad < ad + bc$
$\therefore\ 2ad < ad + bc$

Step 1: show $(ad + bc < 2bc)$:
if $(x < z)$
$\therefore\ 2ad < 2bc$
$\therefore\ ad < bc$
$\therefore\ ad + bc < bc + bc$
$\therefore\ ad + bc < 2bc$

We have shown that for any positive rational numbers x and z, where $x < z$, there exists a positive rational number $y = (x + y)/2$, where $x < y < z$.

Question 3. [5 marks]
average: **3.4** (median: 4, mode: 5)
Re-write the following statement formally, and then prove or disprove the statement:
(You can use the predicate $P(x) \Leftrightarrow x$ is prime).

For any prime numbers x and z, where $1 < x < z$, $(x + z)$ is not prime.

SOLUTION

$$\forall x \in \mathbb{Z}^+, \forall z \in \mathbb{Z}^+, (P(x) \wedge P(z) \wedge (1 < x < z)) \rightarrow \sim P(x + z)$$

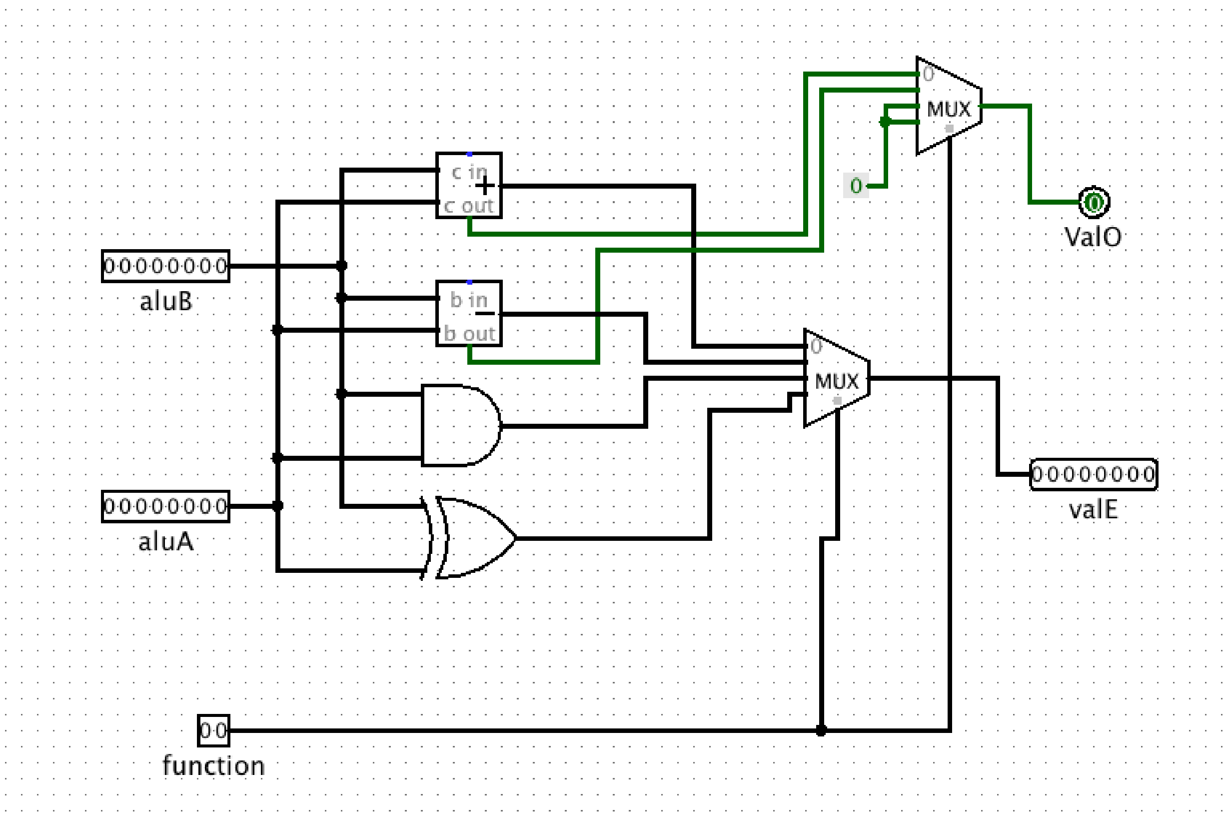
The statement is false by counterexample: select $x = 2$ and $z = 3$:

$P(2) \equiv T$
$P(3) \equiv T$
$P(5) \equiv T$

$(P(x) \wedge P(z) \wedge (1 < x < z)) \rightarrow \sim P(x + z)$
$\equiv (T \wedge T \wedge T) \rightarrow F$
$\equiv F$

Question 4. [10 marks]
average: **6.1** (median: 7, mode: 8)

Given the following circuit from the lab, complete the empty 8 cells in the table:



aluA	aluB	function	valE	val0
00010010	00001011	01		
01010010	10001001	11		
01110010	10001011		00000010	0
11101011	00010110	00		
	00010110	01	00000101	0

SOLUTION

aluA	aluB	function	valE	val0
00010010	00001011	01	11111001	1
01010010	10001001	11	11011011	0
01110010	10001011	10	00000010	0
11101011	00010110	00	00000001	1
00010001	00010110	01	00000101	0

Question 5. [10 marks]
average: **7.2** (median: 7.75, mode: 9)
Re-write the following statement informally (in English), using the simplest language that you can. Then, write the *NEGATION* of the statement formally with all quantifiers at the beginning (on the left) and with no negations in front of (to the left of) any quantifiers. Then prove or disprove the new (negated) statement:

$\forall x \in \mathbb{Z}^+, (\exists k \in \mathbb{Z}^+, x = 2k) \rightarrow \sim(\exists y \in \mathbb{Z}^+, y < x)$

SOLUTION

All even numbers are the smallest number.

$\sim(\forall x \in \mathbb{Z}^+, (\exists k \in \mathbb{Z}^+, x = 2k) \rightarrow \sim(\exists y \in \mathbb{Z}^+, y < x))$
$\equiv \sim(\forall x \in \mathbb{Z}^+, \sim(\exists k \in \mathbb{Z}^+, x = 2k) \vee \sim(\exists y \in \mathbb{Z}^+, y < x))$
$\equiv \sim(\forall x \in \mathbb{Z}^+, (\forall k \in \mathbb{Z}^+, \sim(x = 2k)) \vee (\forall y \in \mathbb{Z}^+, \sim(y < x)))$
$\equiv \sim(\forall x \in \mathbb{Z}^+, \forall k \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, (\sim(x = 2k)) \vee (\sim(y < x)))$
$\equiv \exists x \in \mathbb{Z}^+, \exists k \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, \sim((\sim(x = 2k)) \vee (\sim(y < x)))$
$\equiv \exists x \in \mathbb{Z}^+, \exists k \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, (x = 2k) \wedge (y < x)$

Direct Proof (Existential):

choose $x = 42, k = 21, y = 5, x, k, y \in \mathbb{Z}^+$

$(x = 2k) \wedge (y < x)$
$\equiv T \wedge T$
$\equiv T$

Therefore, the negation (there exists an even number that is not the smallest number) is true

Question 6. [5 marks]
average: **3.4** (median: 4.5, mode: 5)
Using the contradiction proof technique, prove:
if x is irrational, then $1/x$ is irrational.
(partial marks if you use another proof technique)

SOLUTION

original statement: (multiple equivalent representations exist)
$\forall x \in \mathbb{R}, (x \notin \mathbb{Q}) \rightarrow (1/x \notin Q)$

Let's assume that statement is not true: (step by step)
$\sim(\forall x \in \mathbb{R}, (x \notin \mathbb{Q}) \rightarrow (1/x \notin \mathbb{Q}))$
$\equiv \exists x \in \mathbb{R}, \sim((x \notin \mathbb{Q}) \rightarrow (1/x \notin \mathbb{Q}))$
$\equiv \exists x \in \mathbb{R}, \sim(\sim(x \notin \mathbb{Q}) \vee (1/x \notin \mathbb{Q}))$
$\equiv \exists x \in \mathbb{R}, (x \notin \mathbb{Q}) \wedge (1/x \in \mathbb{Q})$

Premise: there exists an irrational number (x) where its reciprocal $(1/x)$ is rational:

1) x is irrational (from premise)
2) $\therefore\ x \neq 0$ (from 1, 0 is rational) *(not necessary, but avoids confusion)*
3) $1/x$ is rational (from premise)
4) $\therefore\ 1/x = a/b$, where a, b are integers. (from 3, def'n of rational)
5) $\therefore\ x = b/a$ (from 4)
6) $\therefore\ x$ is rational (from 4,5, def'n of rational)

So we have reached a contradiction ($x \in \mathbb{Q}, x \notin \mathbb{Q}$), therefore negated statement is false, and original statement is true.

Question 7. [1 mark]
Please respond as honestly as you can: (circle one per row)

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree	
This exam was too long	1	2	3	4	5	average: **3.0**
This exam was too hard	1	2	3	4	5	average: **3.2**
The exam content was fair	1	2	3	4	5	average: **3.6**

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