IMPORTANT FIRST STEPS:
1. Form a group of 2 students.
2. Clearly put your names and IDs on 1 copy of this worksheet.
3. Be sure to turn this exercise in at the end of class.
4. Do not attempt to prove the inductive step until you have completed all other steps for each question.

Mathematical Induction

1. Using mathematical induction, prove the following:

$$1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1) / 6$$

S(n): $1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1) / 6$

Claim: $\forall n \geq 0$, $S(n)$ holds.

Base Case, express and prove:

n=1
S(n) = 1 = 1(1+1)(2(1)+1) / 6

Inductive Hypothesis: Assume S(n) that is $1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1) / 6$

What is to be proven using the Inductive Step? Write this out explicitly:

Prove S(n+1) is also true: $1^2+2^2+3^2+...+(n+1)^2 = (n+1)((n+1) +1)(2 (n+1) +1) / 6$

Now, prove the Inductive Step:

We must prove that: $1^2+ 2^2+ ... +n^2+ (n+1)^2 = (n+1) [(n+1) + 1] [2(n+1) +1] / 6$

$1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1) / 6$, by the inductive hypothesis
Now add $(n+1)^2$ to both sides and then show the algebra to prove that the two sides are equal

Statement that you've proven your claim:

Therefore, by the principle of Mathematical Induction, S(n) holds for all values of n.

QED.

2. Using mathematical induction, prove the following:

$$1^3+2^3+3^3+...+n^3 = n^2(n+1)^2 / 4$$

S(n): $1^3+2^3+3^3+...+n^3 = n^2(n+1)^2 / 4$

Claim: $\forall n \geq 0$, $S(n)$ holds.

Base Case, express and prove:

n=1
$S(n) = 1 = 1^2(1+1)^2 / 4$

Inductive Hypothesis:

Assume S(n), that is $1^2+2^2+3^2+...+n^2 = n^2(n+1)^2/4$

What is to be proven using the Inductive Step? Write this out explicitly:

Prove S(n+1) is true: $1^3+ 2^3+ ... +n^3+ (n+1)^3 = (n+1)^2 [(n+1) + 1]^2 / 4$

Now, prove the Inductive Step:

We must prove that: $1^3+ 2^3+ ... +n^3+ (n+1)^3 = (n+1)^2 [(n+1) + 1]^2 / 4$

$1^2+2^2+3^2+...+n^2 = n^2(n+1)^2/4$, by the inductive hypothesis

Now, add $(n+1)^3$ to both sides and then show the algebra to prove that both sides are equal

Statement that you've proven your claim:

Therefore, by the principle of Mathematical Induction, S(n) holds for all values of n.

3. Using mathematical induction, prove that the the distributed property of OR can be extended to a disjunction with a conjunction of *n* sentences. I.e.

$$(a \vee (b \wedge c \wedge \ldots \wedge n)) = (a \mathbf{v} b) \wedge (a \mathbf{v} c) \wedge \ldots \wedge (a \mathbf{v} n)$$

S(n): $(p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_n)) = (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee p_n)$
(Notice how I've switched the notation to p's and subscripted q's...this will make it more precise below.)

Claim:

$\forall n \geq 0$, $S(n)$ holds.

Base Case, express and prove:

n=2
Given the Distribution law, we have the following:
$(p \vee (q_1 \wedge q_2) = (p \vee q_1) \wedge (p \vee q_2)$

Inductive Hypothesis:

Assume S(n), that is $(p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_n)) = (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee p_n)$

What is to be proven using the Inductive Step? Write this out explicitly:

Prove S(n+1) is also true: $(p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_{n+1})) = (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee p_{n+1})$

Now, prove the Inductive Step:

Consider $(p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_{n+1}))$
$= (p \vee [(q_1 \wedge q_2 \wedge \ldots \wedge q_n) \wedge q_{n+1}])$ by the Associative Law
$= (p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_n)) \wedge (p \vee q_{n+1})$ by the Distributive Law
$= (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_n) \wedge (p \vee q_{n+1})$ by the Inductive Hypothesis.

Statement that you've proven your claim:

Therefore, by the principle of Mathematical Induction, S(n) holds for all values of n.

4. Prove that the sum of the interior angles of a convex polygon of n sides is (n-2) * 180 degrees.

S(n): The sum of the interior angles of a convex polygon of n sides is (n-2) * 180 degrees. Note that a closed-sided shape of n sides, will also have m vertices, so n=m.

Claim: $\forall n \geq 3, S(n)$ holds.

Base Case, express and prove: The sum of the angles of a triangle (n=3) is 180 by definition of a triangle.

Inductive Hypothesis: Assume that S(n) is true. That is, there exists some convex polygon, Y, that has vertices $x_1, x_2, \ldots, x_m$, the sum of whose angles is (m-2) * 180. (Remember m=n.)

What is to be proven using the Inductive Step? Write this out explicitly:

We need to show that a convex polygon, X, that has vertices $x_1, x_2, \ldots, x_{m+1}$ has angles that sum to ((n+1)-2) * 180.

Now, prove the Inductive Step:

Consider any two arbitrary, side-by-side vertices within Y. Let's say it's x_m and x_1. Right now, x_m and x_1 each contribute the value of their inside angles to the total sum of the angles in Y. To get to X we need to insert another vertex in Y. Let's put it between x_m and x_1. Now we have x_m, x_{m+1} and x_1. If you connect these three points you'll see that they draw a triangle. We'll call this triangle T. If you consider the size of the angles of x_m and x_1 before we added in x_{m+1}, and then after, you should see that the size of the shape has grown by the size of the triangle T. Thus the sum of the angles of X is in fact the sum of Y, plus the angles of T. Since T is a triangle, we know that contributes 180. So, given the inductive hypothesis, we have the following:

$$((m-2) * 180) + 180 = ((m-2) + 1) * 180 = ((m+1) - 2) * 180$$

Statement that you've proven your claim:

Therefore, by the principle of Mathematical Induction, S(n) holds for all values of n.