Mathematical Induction

1. Postage of *n* cents can be formed using 4-cent and 5-cent stamps.

S(n): Postage of n cents can be formed using 4-cent and 5-cent stamps.

Claim: $\forall n \geq 12$, $S(n)$ holds.

Base Case, express and prove:

$S(12)$ holds... just add three 4-cent stamps together.
$S(13)$ holds... just add two 4-cent stamps together with one 5-cent stamp.
$S(14)$ holds... just add one 4-cent stamp together with two 5-cent stamps.
$S(15)$ holds... just add three 5-cent stamps together.

Inductive Hypothesis: Assume that it is possible to produce n-cents of postage for 12, 13, ..., n cents.

What is to be proven using the Inductive Step? Write this out explicitly:

Prove that it is possible to form n+1 cents of postage using 4- and 5-cent stamps.

Now, prove the Inductive Step:

Take the postage that you used for (n-3) stamps and add a 4-cent stamp. Now, you have (n+1) cents worth of postage.

Statement that you've proven your claim:

Therefore, by the principle of Strong Mathematical Induction, S(n) holds for all values of n.

2. What's wrong with this proof? Anything?

Theorem: All horses are the same colour.

Proof:

Base case: All horses in any group of one horse are obviously the same colour.

Induction hypothesis: Assume that all horses in any group of size n are the same colour (for an arbitrary integer $n \geq 1$).

Inductive step: Under this assumption, we need to prove that all horses in any group of horses of size n+1 are the same colour.

Consider an arbitrary group of n+1 horses.

Remove any one horse from it. What remains is a group of n horses, which are all the same colour by the induction hypothesis. Only the set-aside horse may be a different colour.

Now, return the horse to the group and remove a different horse. Again, the remaining horses are all the same colour, but from the previous step we already know that this time the set-aside horse is also the same colour. Therefore, all horses in any group of size n+1 are the same colour.

This concludes our inductive proof. Therefore all horses in any group of any size are the same colour, i.e., all horses are the same colour.

The inductive step here is faulty because it's relying on the fact that the n=1 and n=2 cases were proven as base cases, therefore this proof is invalid. Obviously, we can't prove that any group of 2 horses is the same colour, so it wouldn't be possible to fix this proof.

3. What's wrong with the following "proof" via strong induction?

All natural numbers are even.

Base case: $n = 0$ is even, so $S(0)$ is true.

Inductive hypothesis: $S(n)$ is even.

Inductive step:

$\forall n \geq 0$, we assume $S(0), S(1), \ldots, S(n)$ and must prove $S(n + 1)$, i.e., assume that $0, 1, \ldots, n$ are even. Consider $n + 1$. $S(n)$ tells us that n is even, and $S(1)$ tells us that 1 is even.

Thus, $n + 1$ is even, since a sum of two even numbers is even. Thus $S(n + 1)$ is true.

By strong induction, we have concluded that $S(n)$ is true for all natural numbers.

P(0) is true, so the base case is true. For the inductive step, this proof shows that P(1) implies P(2), P(1) and P(2) -> P(3), and that for all n ≥ 1, P(1),...,P(n) -> P(n+1). But, we don't have the key "domino", namely that P(0) -> P (1). In other words, when n = 0, we are making a circular argument by assuming P (1) is true in order to prove that P (1) is true! In this case, P (1) is false, so the induction fails.