CPSC 121 Quiz 5

Wednesday, 2012 Aug 1

Name: ______________________ Student ID: ______________________

Signature: ______________________

Your signature acknowledges your understanding of and agreement to the rules below.

– You have 30 minutes to write the 2 questions on this examination. A total of 20 marks are available.

– You may have as an aide up to 3 textbooks and a 3 inch stack of paper notes and nothing else. **No electronic devices allowed**; so, no cell phones and no calculators.

– Keep your answers short. If you run out of space for a question, you have likely written too much.

– The number in square brackets to the left of the question number indicates the number of marks allocated for that question. Use these to help you plan your use of time on the exam.

– Clearly indicate your answer to each problem. If your answer is not in the provided blank, then indicate where the answer is, and at the answer's location indicate the question it addresses.

– **Good luck!**

| Question | Marks |
|---|---|
| 1 | |
| 2 | |
| Total | |

UNIVERSITY REGULATIONS:

– Each candidate should be prepared to produce, upon request, his/her UBC card.

– No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

– CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
2. Speaking or communicating with other candidates.
3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

– Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

[8] 1. We are planning a proof by induction on $b$ that: we can compute $a^b$ for any $a \in \mathbb{Z}, b \in \mathbb{Z}^+$ in at most $2\log_2 b$ multiplications.

Recall: $\mathbb{Z}$ is the integers. $\mathbb{N}$ is the natural numbers $(0, 1, 2, 3, \ldots)$, $\lfloor x \rfloor$ is the largest integer $y \leq x$.

[2] (a) We will always have at least one base case in our inductive proofs. Which base case will we want on this proof, regardless of how the proof proceeds? (Circle one.)

i. $b = 0$
ii. $b = 1$
iii. $b = 2$
iv. $b = 3$

[2] (b) Say that the inductive step of our proof says that for any "sufficiently large" $b$: *we can calculate $a^b$ with at most $2\log_2 b$ multiplications by multiplying $a^{\lfloor b/2 \rfloor}$ by itself and, if $b$ is odd, multiplying the result by $a$. . . .*

We already have one base case from above. What *additional* base case(s), if any, do we need? (For full credit, your answer should be consistent with your previous answers. **Circle *all* that apply.**)

i. $b = 0$
ii. $b = 1$
iii. $b = 2$
iv. $b = 3$
v. No additional base case.

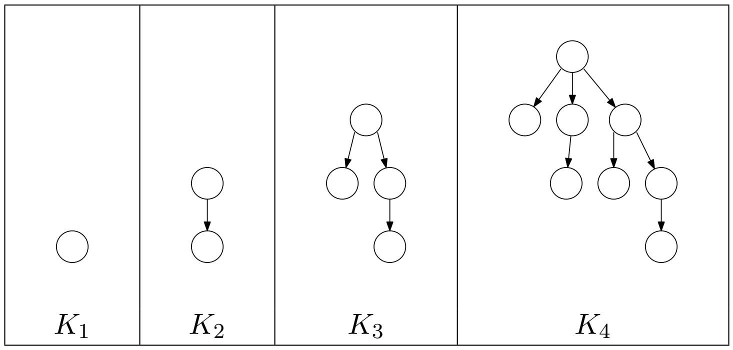
[2] (c) Continuing the example, we should quantify what we mean by "sufficiently large". How large should $b$ be for the **inductive step** of the proof by induction? (For full credit, your answer should be consistent with your previous answers. Circle one.)

i. $b \geq 0$
ii. $b \geq 1$
iii. $b \geq 2$
iv. $b \geq 3$

[2] (d) Continuing the example, what should our induction hypothesis be? (Circle one.)

i. "we can compute $a^b$ in at most $2\log_2 b$ multiplications"
ii. "we can compute $a^{b-1}$ in at most $2\log_2(b-1)$ multiplications"
iii. "we can compute $a^{\lfloor b/2 \rfloor - 1}$ in at most $2\log_2(\lfloor b/2 \rfloor - 1)$ multiplications"
iv. "we can compute $a^{\lfloor b/2 \rfloor}$ in at most $2\log_2 \lfloor b/2 \rfloor$ multiplications"

[12] 2. We create a new kind of tree called a "kitchen sink tree" (or KST) in which each node can have any number of subtrees. The first KST tree $K_1$ is a single node with no children. The second KST tree $K_2$ has $K_1$ as its only child. $K_3$ has $K_1$ and $K_2$ as children. $K_4$ has all of $K_1$, $K_2$, and $K_3$ as children. Here are these first KST trees:




In general, $K_n$ has all previous $K_i$ trees as its children.

Prove by induction that $K_n$ has $2^{n-1}$ nodes for all integers $n \geq 1$. You may assume that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$.

(You need not prove that your proof structure terminates.)

**BONUS:** Earn up to 2 bonus points by doing one or more of these problems.

- How does (1) the number of multiplications needed on $n$ for the technique we used for insight in the first problem relate to (2) the binary representation of $n$? Why?
- There are (at least) two *radically different* inductive proofs of the second problem. Give the other one. (Be careful to clearly establish any facts you need.)