# CPSC 121 Quiz 5
Wednesday, 2012 Aug 1

[8] 1. We are planning a proof by induction on $b$ that: we can compute $a^b$ for any $a \in \mathbb{Z}, b \in \mathbb{Z}^+$ in at most $2 \log_2 b$ multiplications.

Recall: $\mathbb{Z}$ is the integers. $\mathbb{N}$ is the natural numbers $(0, 1, 2, 3, \ldots)$, $\lfloor x \rfloor$ is the largest integer $y \leq x$.

[2] (a) We will always have at least one base case in our inductive proofs. Which base case will we want on this proof, regardless of how the proof proceeds? (Circle one.)

i. $b = 0$
ii. $b = 1$
iii. $b = 2$
iv. $b = 3$

**Solution:** $b \in \mathbb{Z}^+$, and the smallest element of $\mathbb{Z}^+$ is 1; so, it's $b = 1$.

(Why no $a$ case? We'll treat $a$ with a WLOG strategy here. It doesn't play a part in the induction.)

[2] (b) Say that the inductive step of our proof says that for any "sufficiently large" $b$: *we can calculate $a^b$ with at most $2 \log_2 b$ multiplications by multiplying $a^{\lfloor b/2 \rfloor}$ by itself and, if $b$ is odd, multiplying the result by $a$. . . .*

We already have one base case from above. What *additional* base case(s), if any, do we need? (For full credit, your answer should be consistent with your previous answers. **Circle *all* that apply.**)

i. $b = 0$
ii. $b = 1$
iii. $b = 2$
iv. $b = 3$
v. No additional base case.

**Solution:** Well, let's test. Does $b = 2$ work?

We need the $\lfloor b/2 \rfloor$ case, according to our insight. That's $\lfloor 2/2 \rfloor = 1$, which we've already handled. So, we don't need $b = 2$. $b = 3$ works similarly, and all the subsequent cases are also handled correctly.

Thus, "No additional base case".

(Note: we never took off credit for you circling base case(s) you already had. . . but you shouldn't do so. We're asking for *additional* base cases here!)

Note that if you chose $b = 0$ on the previous problem, you'd choose "No additional base case" here. (The 1 and 2 and all subsequent cases all break down in terms of the 1 case.) If you chose $b = 2$, then you also need $b = 3$ here. All subsequent cases break down in terms of $b = 2$ and $b = 3$. If you chose $b = 3$ above, hopefully you realized here that you were wrong, since no answer is right! (But, we allowed you to cross out all the answers and write in the correct one, which is $b = 4$ and $b = 5$.)

[2] (c) Continuing the example, we should quantify what we mean by "sufficiently large". How large should $b$ be for the **inductive step** of the proof by induction? (For full credit, your answer should be consistent with your previous answers. Circle one.)

i. $b \geq 0$
ii. $b \geq 1$
iii. $b \geq 2$
iv. $b \geq 3$

**Solution:** We want to "stay away from" our base cases. So, we want $b \geq 2$.

Generally speaking, you need $b \geq k+1$, where $k$ is your largest base case that you've chosen so far. (Again, if you chose $b = 3$, you're out of luck unless you cross out all the answers and write in your own. Sorry!)

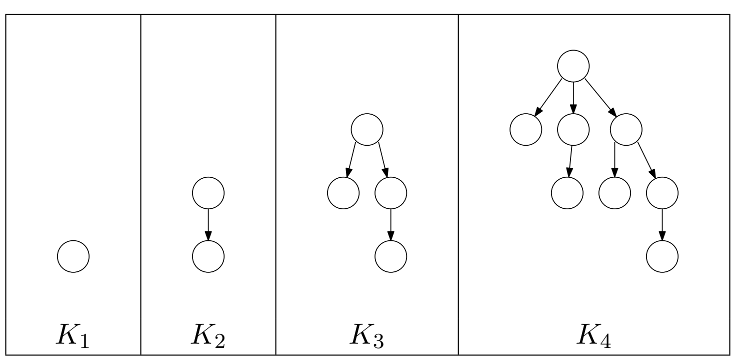
[2] (d) Continuing the example, what should our induction hypothesis be? (Circle one.)

i. "we can compute $a^b$ in at most $2\log_2 b$ multiplications"
ii. "we can compute $a^{b-1}$ in at most $2\log_2(b-1)$ multiplications"
iii. "we can compute $a^{\lfloor b/2 \rfloor - 1}$ in at most $2\log_2(\lfloor b/2 \rfloor - 1)$ multiplications"
iv. "we can compute $a^{\lfloor b/2 \rfloor}$ in at most $2\log_2\lfloor b/2 \rfloor$ multiplications"

**Solution:** We need to assume that the *cases we need* work correctly. We only need the $\lfloor b/2 \rfloor$ case here. So, it's: "we can compute $a^{\lfloor b/2 \rfloor}$ in at most $2\log_2\lfloor b/2 \rfloor$ multiplications".

Note that nothing in this problem required you to know anything about logs. That said, it's a good exercise to run off and *prove* this now!

[12] 2. We create a new kind of tree called a "kitchen sink tree" (or KST) in which each node can have any number of subtrees. The first KST tree $K_1$ is a single node with no children. The second KST tree $K_2$ has $K_1$ as its only child. $K_3$ has $K_1$ and $K_2$ as children. $K_4$ has all of $K_1$, $K_2$, and $K_3$ as children. Here are these first KST trees:




In general, $K_n$ has all previous $K_i$ trees as its children.

Prove by induction that $K_n$ has $2^{n-1}$ nodes for all integers $n \geq 1$. You may assume that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$.

(You need not prove that your proof structure terminates.)

**Solution :** NOTE: it is possible to prove this with another approach, for which we gave full credit if done *carefully*!

Now, the insight here is that $K_n$ is made up of the subtrees $K_1$, $K_2$, …, $K_{n-1}$ all hanging from the one additional root node. That gives us all the structure we need!

**Proof Sketch**: By induction on $n$ (which KST we're dealing with).

**Base Case**: *establish that $K_1$ "works correctly", i.e., $K_1$ has $2^{1-1}$ nodes; $K_2$ should work fine based on $K_1$ using our insight; so, no need for more base cases, but double-check!*

**Induction Hypothesis**: For an arbitrary integer $n > 1$, assume that all $K_i$ for $1 \leq i < n$ have $2^{i-1}$ nodes.

**Inductive Step**: Since $n > 1$, $K_n$ is a node with every previous $K_i$ as a child. Thus, its number of nodes is… *Figure this out, at some point using our IH, which allows us to change "number of nodes in $K_i$" into $2^{i-1}$ for all of those "previous $K_i$"; we want to end up with $2^{n-1}$.*

**Termination (not required)**: For a finite positive integer $n$, the inductive step proceeds based on values ranging from 1 up to $n-1$; so, never smaller than 1 and always at least 1 smaller than $n$. Thus, in a finite number of steps, we eventually reach the base case of 1 (along every path in the inductive step).

QED

Now we're ready to go with the actual proof.

**Proof Sketch**: By induction on $n$ (which KST we're dealing with).

**Base Case**: $K_1$ has a single node (as given). And, $2^{1-1} = 2^0 = 1$. ✓

**Induction Hypothesis**: For an arbitrary integer $n > 1$, assume that all $K_i$ for $1 \leq i < n$ have $2^{i-1}$ nodes.

**Inductive Step**: Since $n > 1$, $K_n$ is a node with every previous $K_i$ as a child. Thus, its number of nodes is $1 + \sum_{i=1}^{n-1}(\text{nodes in } K_i)$. (That's the top node plus all the children's nodes.) By the IH, this is $1 + \sum_{i=1}^{n-1} 2^{i-1} = 1 + \sum_{j=0}^{n-2} 2^i = 1 + (2^{n-2+1} - 1) = 2^{n-1}$.

**Termination (not required)**: For a finite positive integer $n$, the inductive step proceeds based on values ranging from 1 up to $n-1$; so, never smaller than 1 and always at least 1 smaller than $n$. Thus, in a finite number of steps, we eventually reach the base case of 1 (along every path in the inductive step).

QED

**BONUS:** Earn up to 2 bonus points by doing one or more of these problems.

- How does (1) the number of multiplications needed on $n$ for the technique we used for insight in the first problem relate to (2) the binary representation of $n$? Why?

  **Solution :** Left as an exercise, but notice that $a^6$ requires one multiplication to get to $a^3$, but then 2 to get to $a^1$, which also requires two multiplications. Why the extra multiplications? Because the last two numbers are odd. How many times will we hit an "odd" case? Well... maybe that has something to do with the binary representation of the number!

- There are (at least) two *radically different* inductive proofs of the second problem. Give the other one. (Be careful to clearly establish any facts you need.)

  **Solution :** Left as an exercise, but consider what happens to $K_n$ (for $n > 1$) if we "chop off" its $K_{n-1}$ child. What does it look like—and, in fact, is it—then?