CPSC 121 Quiz 4
Wednesday, 2012 July 25

[2] 1. In proving the following theorem with direct proof techniques, you would choose values for a and b. Which of p, q, and r can your choices for a's and b's values depend on? (For example, if all of these were integers and b could be chosen to be $p * q + r$, then it can depend on all of p, q, and r.)

Theorem:

$$\exists a \in A, \forall p \in B, \forall q \in A, R(a,p,q) \rightarrow \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r)$$

Solution : a's value can depend on none of these.

Solution : b's value can depend on p and q (but not r), since it is only inside the quantifiers for p and q. (Its value can also depend on a's value.)

[2] 2. Imagine you were instead going to prove the following theorem by contradiction (as your first step). Write the appropriate assumption with any negations moved "inward" as much as possible. When you're done, negation(s) should appear only on predicates—e.g., $\sim M(a,b)$ or $b \neq c$—and not on quantifiers or parenthesized expressions.

You need not show each step of moving negations inward (but you can, if you wish!).

Theorem:

$$\exists a \in A, \forall p \in B, \forall q \in A, R(a,p,q) \rightarrow \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r)$$

Solution : The assumption is the negation of the entire theorem. We'll do it step-by-step.

$\sim \exists a \in A, \forall p \in B, \forall q \in A, R(a,p,q) \rightarrow \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r)$

$\forall a \in A, \sim \forall p \in B, \forall q \in A, R(a,p,q) \rightarrow \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r)$ by GDM

$\forall a \in A, \exists p \in B, \sim \forall q \in A, R(a,p,q) \rightarrow \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r)$ by GDM

$\forall a \in A, \exists p \in B, \exists q \in A, \sim (R(a,p,q) \rightarrow \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r))$ by GDM

$\forall a \in A, \exists p \in B, \exists q \in A, \sim (\sim R(a,p,q) \vee \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r))$ by IMP

$\forall a \in A, \exists p \in B, \exists q \in A, R(a,p,q) \wedge \sim \exists b \in B, N(a,b) \wedge \forall r \in B, P(q,b,a,r)$ by DM

$\forall a \in A, \exists p \in B, \exists q \in A, R(a,p,q) \wedge \forall b \in B, \sim (N(a,b) \wedge \forall r \in B, P(q,b,a,r))$ by GDM

$\forall a \in A, \exists p \in B, \exists q \in A, R(a,p,q) \wedge \forall b \in B, \sim N(a,b) \vee \sim \forall r \in B, P(q,b,a,r))$ by DM

$\forall a \in A, \exists p \in B, \exists q \in A, R(a,p,q) \wedge \forall b \in B, \sim N(a,b) \vee \exists r \in B, \sim P(q,b,a,r))$ by GDM

[4] 3. For each of the following theorems, indicate the most complete "direct" proof approach—no proof by contradiction, no use of logical equivalences—that you can by writing the letters of the techniques listed below in the order you would use them. Ignore any blanks you don't need.

List of proof techniques:

A. Witness proof (in Epp: constructive proof of existence)

B. "Without loss of generality" proof (in Epp: generalizing from the generic particular)

C. Antecedent assumption.

D. Equality or inequality proof.

Theorem: $\exists x \in S, (x \geq 5) \rightarrow \forall y \in S, P(x, y)$.

Solution : #1 A, #2 C, #3 B

We assume $x \geq 5$. We do *not* prove it, and so we do not need the inequality proof technique.

Theorem: $\forall v_1 \in \mathbb{R}^+, \forall v_2 \in \mathbb{R}^-, \exists c \in \mathbb{R}, v_1 * v_2 \leq c$

Solution : #1 B, #2 B, #3 A, #4 D

[3] 4. While cleaning a robot purchased for your parents' farm, you accidentally trigger a damaged recording of a human-like female alien proving some theorem.

Below is the remaining text of the proof. Write out the theorem **in as much detail as possible**. (Use the predicates Old and $Green$ if they help.)

> Without loss of generality, let y be a member of J. We proceed in two cases. Case 1: Assume y is old. Then, let $p =$ QG (also a member of J). We now show that $mc(p) < mc(y)$. [*part of the proof is impossible to hear*] Case 2: Assume y is green. Then, let $p =$ LS (also a member of J). We now show that $mc(p) < mc(y)$. [*and the recording ends*]

Solution : Here's a step-by-step breakdown:

Without loss of generality, let y be a member of J... looks like $\forall y \in J$.

We proceed in two cases... sounds like an "OR" coming up.

Case 1... is the first disjunct

Assume y is old... restricts the domain of the universal above $Old(y) \rightarrow \ldots$

Then, let $p =$ QG (also a member of J)... is a choice by the prover; so, an existential: $\exists p \in J$.[1]

We now show that $mc(p) < mc(y)$... Sounds like the rest of the statement. Thus, we so far have $\forall y \in J, (Old(y) \rightarrow \exists p \in J, mc(p) < mc(y)) \vee \ldots$.

[1] Note that QG doesn't (and probably shouldn't) show up. If it did, why bother with the existential (when you could just use the constant QG directly)? Shifting to an entirely silly tone: Why *force* the issue? Seems like be*knight*ed thinking for a *master* of logic. We don't *sense* that saving space by leaving it out is important... and *yet I* can't help saying that it is a *light saver of space*.

Case 2. . . is the second disjunct

Assume y **is green. . .** as above, restricts the domain of the universal $Green(y) \rightarrow \ldots$

Then, let $p =$ **LS (also a member of** J**)** as above, indicates an existential: $\exists p \in J$

We now show that $mc(p) < mc(y)$**. . .** is again the same as above (and again ends this case). So, we know have the whole answer.

$$\forall y \in J, (Old(y) \rightarrow \exists p \in J, mc(p) < mc(y)) \vee (Green(y) \rightarrow \exists p \in J, mc(p) < mc(y))$$

Just to be fancy, we note that this was likely originally stated as:

$$\forall y \in J, (Old(y) \vee Green(y)) \rightarrow \exists p \in J, mc(p) < mc(y)$$

[9] 5. Consider the following theorem: for every integer $b \geq 2$, there are integers k and n with $0 < k < n$ such that $\frac{k}{n}$ is representable in base b with a finite number of digits.

[2] a. Translate the theorem to predicate logic. (Use $R(a, b)$ to mean "the fraction a is representable in base b with a finite number of digits.")

Solution :

Theorem: $\forall b \in \mathbb{Z}, b \geq 2 \rightarrow \exists k \in \mathbb{Z}, \exists n \in \mathbb{Z}, 0 < k < n \wedge R(\dfrac{k}{n}, b)$

[7] b. Prove the theorem. Use a direct proof, making the structure of your proof **clear**. (Partial credit is available for getting the structure correct.)

Solution : Scratch work from the structure:

WLOG, let b be an arbitrary integer. Assume $b \geq 2$. Let $k = \underline{??}, n = \underline{??}$ [note: can be based on b; must be integers; the inequality $0 < k < n$ must be true]. Show that $\frac{k}{n}$ is representable in base b.

Well, let's pick some simple values. $\frac{1}{2}$ is representable in base 2 but not in base 3. But, $\frac{1}{3}$ is representable in base 3. $\frac{1}{5}$ isn't representable in either of them, but it *is* in base 5. Aha! Let's try $\frac{1}{b}$!

Now, on to. . .

Proof: WLOG, let b be an arbitrary integer. Assume $b \geq 2$. Let $k = 1, n = b$. (Note that $0 < 1 < b$ since $b \geq 2$.) The fraction $\frac{1}{b}$ will be 0.1 in base b, regardless of the value of b. (Since the place to the right of the "decimal" point is the b^{-1} place.)

QED

BONUS: Earn up to 2 bonus points by doing one or more of these problems.

- Translate and prove this theorem: "For every integer $b \geq 2$ and integer k, there is an integer n with $0 < k < n$ such that $\frac{k}{n}$ is representable in base b with a finite number of digits."

Solution : The tricky part here isn't really the proof. It's the translation. How do we meaningfully translate "For every integer $b \geq 2$ and integer k, there is an integer n with $0 < k < n$ such that…"?

This does *not* work:

$\forall b \in \mathbb{Z}, b \geq 2 \rightarrow \forall k \in \mathbb{Z}, \exists n \in \mathbb{Z}, 0 < k < n \rightarrow \ldots$

Can you see why not? There's a hint in the footnote[2].

This *also* does *not* work:

$\forall b \in \mathbb{Z}, b \geq 2 \rightarrow \forall k \in \mathbb{Z}, \exists n \in \mathbb{Z}, 0 < k < n \wedge \ldots$

Again, can you see why not? There's a hint in the footnote[3].

This works:

$\forall b \in \mathbb{Z}, b \geq 2 \rightarrow \forall k \in \mathbb{Z}, 0 < k \rightarrow \exists n \in \mathbb{Z}, k < n \wedge \ldots$

Can you see why? Why can we use the existential restricting idiom even though k is universally quantified? There's a hint in the footnote[4].

For the proof itself, think about the smallest b^i for some integer that's larger than k.

- Translate and prove this other similar theorem: "For every integer $b \geq 2$, there are integers k and n with $0 < k < n$ such that $\frac{k}{n}$ is *not* representable in base b with a finite number of digits."

Solution : Here the translation's trivial, but what of the proof?

There are many reasonable approaches. We like the one hinted at by this: We saw on a previous quiz that $\frac{1}{2}$ is $0.1111\ldots$ in base 3. We also know in base 10 that $\frac{1}{9}$ is $0.1111\ldots$. It turns out that $\frac{1}{1} = 1$ *can* be represented as $0.1111\ldots$ in base 2, although that's exactly equal to 1 (just as $0.9999\ldots$ in base 10 is equal to 1).

BONUS? Maybe worth a small amount of bonus points: In the robot message problem, tell us who the robot and the human alien are and what y, J, p, QG, the $mc(\cdot)$ function, and LS are.

Solution : You might want to check out the footnote to the solution for the original problem. Be warned: you may *sense a disturbing farce*.

[2] $n = 0$

[3] $k = 0$

[4] Is it possible, while constructing a counterexample, to choose k to be $n + 1$? It's not. Why not?