CPSC 121,
2005/6 Winter Term 2, Section 203
Quiz 2, **now in Worksheet Form**!

Name: ________________________________ Student ID: ___________________________

[15] 1. Jubjubs have a height and a size. Jubjubs are formed in three ways:

(1) A new Jubjub of height 1 and size 1 appears every afternoon.

(2) Two Jubjubs of different heights can form a new Jubjub. The new Jubjub's size is the sum of the two parents' sizes and its height is the height of the taller parent.
NOTE: one of the two parents must be the same height as the child!

(3) Two Jubjubs of the same height can form a new, taller Jubjub. The new Jubjub's size is the sum of the two parents' sizes and its height is one larger than its parents' height.
NOTE: one of the two parents must be ≤ half the size of the child!

For example:

- Using (2), a Jubjub of height 5 and size 41 and a Jubjub of height 3 and size 9 can form a Jubjub of height 5 and size 50.
- Using (3), a Jubjub of height 3 and size 20 and a Jubjub of height 3 and size 8 can form a Jubjub of height 4 and size 28.

Using induction on the size of a Jubjub, prove the following for every Jubjub, where **h** is the Jubjub's height and **s** is the Jubjub's size:

$$h \leq (\log_2 s) + 1$$

Help with logs: $\log_2 1 = 0$. $(\log_2 n) + 1 = \log_2 (2n)$.
Hint: proceed in two cases for your inductive step based on how the Jubjub was formed!

On what quantity should we perform induction? __________

BASE CASE
What's the predicate we want to prove for all **n**? **P(n)** ≡ ________________

What do we need to prove in the base case? ______________________

Prove it!

INDUCTIVE STEP (note: I usually write the IS *before* the BC!)

Should we use strong induction or weak induction? (Hint: the answer is **who cares** at this point!)

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For either strong induction or weak induction, we're going to make some assumption about **`P`** being true for some value(s) smaller than an arbitrary **`n`**. Based on those (completely unfounded) assumptions, we'll prove **`P(n)`**. What's **`P(n)`**?

`P(n)` ≡ ___________________________________

In **any** induction problem, start with the thing you're trying to prove, and break it down into "smaller" pieces.

Where does any Jubjub of size **`n > 1`** come from? ______________________

How do we know **`n > 1`** in the inductive step? ___________________

How should we break down into two cases? ________________________

For case 1, ASSUME the parents are different heights.

Which parent is the taller one? Does it matter? ____________________

How tall is the child in comparison to the taller parent? ___________________

How big is the child in comparison to the taller parent? ____________________

Do you have a formula that relates the taller parent's size to its height? ________________
(Hint: you do if you want to. You've broken the problem into smaller pieces; so, **use induction!**)

Prove **`P(n)`** based on knowing **`P(k)`** for all **`k < n`**!

Now, you're done with case 1. Does that prove the whole inductive step? ______________

For case 2, ASSUME the parents are the same height.

Which parent is the smaller one? Does it matter? ____________________
(Note: if they're the same size, let's just pick one and call it the "smaller" one.)

How tall is the child in comparison to the smaller parent? ____________________

How big is the child in comparison to the smaller parent? ____________________

Do you have a formula that relates the smaller parent's size to its height? ________________
(Hint: you do if you want to. You've broken the problem into smaller pieces; so, **use induction!**)

Move the "+ 1" in the smaller parent's height inside the log.

Prove **`P(n)`** based on knowing **`P(k)`** for all **`k < n`**!

Now, you're done with case 2. Does that prove the whole inductive step? ______________

Why can't there be **any** other cases? _______________________

CHALLENGE: why **can't** induction on height work? _________

CHALLENGE: how do we know that if the parents are the same height, one of them has size ≤ half of the child's size?

[10] 2. Prove the following for arbitrary sets A, B, and C: $(A \cap B \subseteq C) \rightarrow B \subseteq \overline{A} \cup C$. Your proof may be in a combination of English and logic but make sure that it's clear what the steps of your proof are and how each step follows from the previous steps.

Where do the parentheses go in $(A \cap B \subseteq C) \rightarrow B \subseteq \overline{A} \cup C$? Put them ALL in.

What type of thing is "$B \subseteq C$"? ____________________
Can you apply intersection to that type of thing? _________________

What type of thing is "B"? ___________________
Can that be the consequent of a conditional ($? \rightarrow B$)? _________________

What does "$(A \cap B \subseteq C) \rightarrow B$" mean? ___________________________________

How do you read $A \cap B \subseteq C$ in English? ______________________

Why isn't that the same as "A intersect B implies C"? ______________________

Why isn't that the same as "A and B are subsets of C"? ______________________

Why **is** that the same as "any x that is an element of A and of B must be an element of C"?

____

How do you write $A \cap B \subseteq C$ as a logical statement about membership?

____

How do you write $B \subseteq A \cup C$ as a logical statement about membership?

____

How do you write $(A \cap B \subseteq C) \rightarrow B \subseteq \overline{A} \cup C$ as a logical statement about membership?

____

Explain why you CANNOT draw a single Venn Diagram for $(A \cap B \subseteq C) \rightarrow B \subseteq \overline{A} \cup C$.

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Draw Venn Diagrams for $(A \cap B \subseteq C)$ and $B \subseteq \overline{A} \cup C$.

Does that prove the theorem? Why or why not? ________________________________

Prove the theorem by:

- translate the whole statement to a logical statement about membership
- assume the left side of the conditional (WHY CAN YOU DO THIS??)
- convert the conditional to a disjunction
- mess around with it until you can convert it back into a conditional that looks like the right side

Prove the theorem by:

- translate the whole statement to a logical statement about membership
- write a truth table that shows the statement is true

(BTW, why do you think no one did this? How much thought does it take?)

Prove the theorem by:

- assume the left side (WHY CAN YOU DO THIS??)
- show that everything in B must either be in A intersect B or outside of A
- show that everything in A intersect B must be in C (based on our assumption)
- state why this proves the theorem!

Prove the theorem by:

- translate the whole statement to a logical statement about membership
- assume the left side of the conditional (WHY CAN YOU DO THIS??)
- prove the contrapositive of the right side by…
  - assume antecedent of the contrapositive
  - show that the consequent of the contrapositive follows

Prove the theorem by CONTRADICTION.

- state the negations of the theorem
- put it in some simplified form
- **prove it!**

[15] 3. Consider the set A = {hello, 3, ∅}, its power set P(A), and its Cartesian product with itself A × A. For each of the following functions, give an example value from the domain and its corresponding image. Then, say whether the function is (**Y**) or is not (**N**) injective, surjective, or bijective. The first row is filled in as an example.

The function `f`	Pre-image	Image	Injective? (**Y** or **N**)	Surjective? (**Y** or **N**)	Bijective? (**Y** or **N**)
`f : A → A` maps values onto themselves: `f(x) = x`.	`hello`	`hello`	Y	Y	Y
`f : A → (A ∪ P(A))` maps values onto themselves: `f(x) = x`.					
`f : A × A → A` picks out the first element from a 2-tuple					
`f : A → P(A)` maps elements into sets: `f(x) = {x}`.					
`f : P(A) → P(A)` "flips" sets: `f(B) = A - B`					
`f` takes 2-tuples of elements from `A` and maps them into subsets of `A`. (**You** pick the particular function.)					

Work this out again… this time, explain why each box is true or false.

CHALLENGE: without knowing what the last function is, which of the injective/surjective/bijective boxes can you answer?

CHALLENGE: which of the injective/surjective/bijective boxes can you answer in general without knowing: (1) what the function is and (2) what `A` is?