Example 7 - Solution

Theorem: $\sqrt{2}$ is an irrational number (i.e. cannot be represented as a/b where $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$)

Proof: We prove this by contradiction. (There isn't any obvious way to do this as a direct proof, and the theorem is not in the form $p \rightarrow q$ so we cannot use the contrapositive.)

Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ where $a \in \mathbb{Z}^+$, $b \in \mathbb{Z}^+$, and (a, b) have no common factor other than 1.

So $(\sqrt{2})^2 = (a/b)^2$

$2b^2 = a^2$

Hence a^2 is even. Therefore, by our last theorem (Example 6), a is even:

$a = 2x$ for some $x \in \mathbb{Z}^+$

So $2b^2 = (2x)^2 = 4x^2$

$b^2 = 2x^2$

which means that b is even.

We have contradicted one of our assumptions. Which one?

… We thus proved that a, b have a common factor (2). This contradicts our assumption that $\sqrt{2}$ is rational.