Practice Midterm 2

This practice exam has five questions. The actual test is shorter. I included more questions on the practice exam to get a more thorough coverage of the examinable topics. The actual midterm is shorter so that you will have time to complete it in one hour.

1. Consider the two languages described below:

$$\begin{aligned} A_1 &= \mathsf{a}^i\mathsf{b}^j\mathsf{c}^c, i < j < k \\ A_2 &= \overline{A_1} \end{aligned}$$

   where i, j, and k integers that are greater than or equal to 0.

   (a) One of these languages is a CFL. Which one? Give a CFG for it.

   (b) One of these languages is not a CFL. Which one? Give a short proof.

2. Consider the CFG with start variable S_0:

$$S_0 \rightarrow \epsilon \mid S_0S_0 \mid S_0\mathsf{a}S_0\mathsf{a}S_0\mathsf{b}S_0 \mid S_0\mathsf{a}S_0\mathsf{b}S_0\mathsf{a}S_0 \mid S_0\mathsf{b}S_0\mathsf{a}S_0\mathsf{a}S_0$$

   (a) Give a one-sentence, English description of the language generated by this grammar?

   (b) Show that this grammar is ambiguous.

   (c) Give an unambiguous grammar for the same language.
   Hint: This language can be recognized by a deterministic PDA.

3. Let B_1 and B_2 two languages.

$$\begin{aligned} \mathit{PerfectShuffle}(B_1, B_2) = \{w \mid & \exists c_1, c_2, \ldots, c_k \in \Sigma.\ \exists d_1, d_2, \ldots, y_d \in \Sigma. \\ & (c_1 \cdot c_2 \cdots c_k \in B_1) \wedge (d_1 \cdot d_2 \cdots d_k \in B_2) \\ & \wedge (w = c_1 \cdot d_1 \cdot c_2 \cdot d_2 \cdots c_k \cdot d_k)\} \end{aligned}$$

   Note that this says that given two strings, x and y, of equal length with $x \in B_1$ and $y \in B_2$, then the string obtained by alternating symbols from x and y is in $\mathit{PerfectShuffle}(B_1, B_2)$. A perfect shuffle corresponds to taking two decks of cards and "shuffling" them so that the composite deck has cards in alternation from the two original decks.

   Are the CFLs closed under $\mathit{PerfectShuffle}$? Give a short proof with your answer.

   If we didn't require perfect alternation, then we would allow the c_i's and d_i's to be strings rather than individual symbols, which would mean that w could have several consecutive symbols from x without an intervening symbol from y and vice-versa. This is called the "*shuffle*" operation that you saw on HW 4, q. ?. The current problem focuses on $\mathit{PerfectShuffle}$ operation which is a close relative to $\mathit{shuffle}$. In particular, $\mathit{PerfectShuffle}(B_1, B_2) \subseteq \mathit{shuffle}(B_1, B_2)$.

4. Let $S = \{u \mid \exists\, \theta \in \mathbb{Q}.\ u = \sin(\theta)\}$. Note that $u \in S$ and $u \neq 0$ then u is irrational. Is the set S countable or uncountable?

5. Consider the two languages described below:

$$\begin{aligned} D_1 &= \{M\#w \mid M \text{ never makes two consecutive moves to the left when reading } w\} \\ D_2 &= \{M\#w \mid M \text{ never makes three consecutive moves to the left when reading } w\} \end{aligned}$$

   (a) One of these languages is Turing decidable. Which one? Give a short justification for your answer.

   (b) One of these languages is not Turing decidable. Which one? Give a short justification for your answer.

   You must give a separate justification for each part. In particular, you can't answer (a) and then say that the other language must be undecidable because the problem stated there was one of each.