CPSC 121 Midterm 1
Friday October 14th, 2016

[12] 1. [1] a. For what truth values of a and b is $a \rightarrow b$ false?

Solution : $a \rightarrow b$ is false only when a is true and b is false.

[3] b. Is $(a \oplus b) \leftrightarrow (a \wedge \sim b) \vee (\sim a \wedge b)$ a tautology, a contradiction, or a contingency? Justify your answer.

Solution : It is a tautology because $a \oplus b$ is logically equivalent to $(a \wedge \sim b) \vee (\sim a \wedge b)$.

[8] c. Using a sequence of logical equivalence, prove that

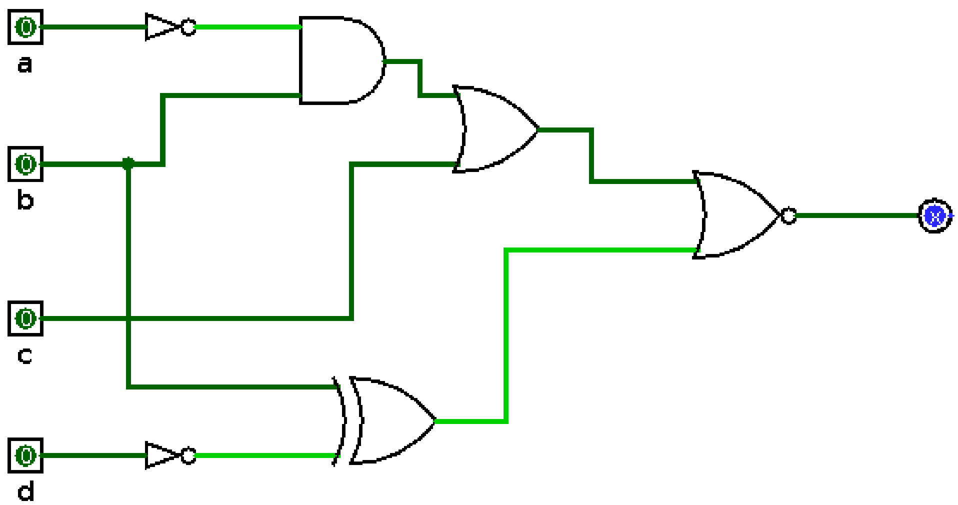
$$(p \oplus q) \vee \sim(\sim p \rightarrow q) \equiv \sim p \vee \sim q$$

Please write the name of the law(s) you applied at each step. Hint: What you showed in part b above may be helpful.

Solution :

LHS	$\equiv (p \wedge \sim q) \vee (\sim p \wedge q) \vee \sim (\sim p \rightarrow q)$	definition of $\oplus$ from part (b)
	$\equiv (p \wedge \sim q) \vee (\sim p \wedge q) \vee \sim (\sim\sim p \vee q)$	definition of $\rightarrow$
	$\equiv (p \wedge \sim q) \vee (\sim p \wedge q) \vee \sim (p \vee q)$	double negative law
	$\equiv (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$	De Morgan's law
	$\equiv (p \wedge \sim q) \vee (\sim p \wedge (q \vee \sim q))$	distributive law
	$\equiv (p \wedge \sim q) \vee (\sim p \wedge T)$	negation law
	$\equiv (p \wedge \sim q) \vee \sim p$	identify law
	$\equiv (p \vee \sim p) \wedge (\sim q \vee \sim p)$	distributive law
	$\equiv T \wedge (\sim q \vee \sim p)$	negation law
	$\equiv \sim q \vee \sim p$	identity law
	$\equiv \sim p \vee \sim q$	commutative law
	$\equiv$ RHS	

[6] 2. [3] a. Consider the following digital circuit. Write a propositional logic expression which is the direct translation of this circuit to propositional logic. Do not simplify your expression.

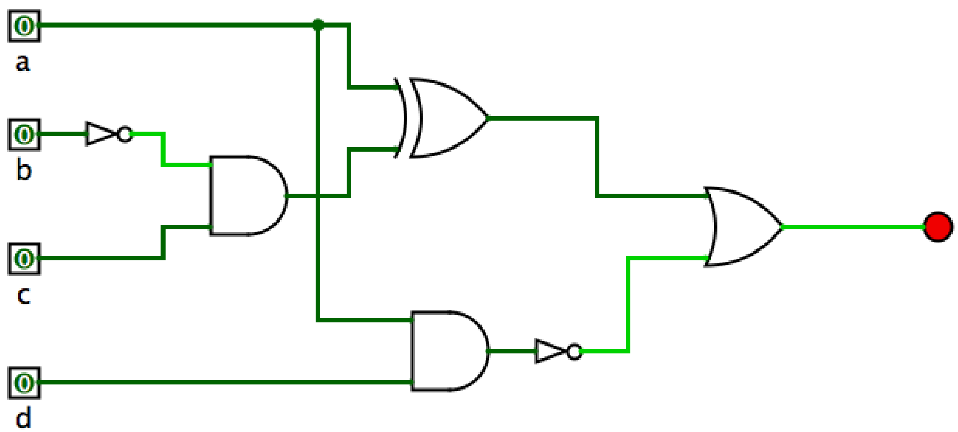


Solution : $\sim (((\sim a \land b) \lor c) \lor (b \oplus \sim d))$

[3] b. Draw the circuit corresponding to the following propositional logic expression.

$$(a \oplus (\sim b \land c)) \lor \sim (a \land d)$$

Solution :



[10] 3. Assume that we use **6 bits** to represented binary integers. If the binary integer is interpreted as a signed integer, then we will represent it using two's complement.

[3] a. Consider the 6-bit binary integer **110101**. If we interpret it as a **signed** binary integer, what is the corresponding decimal value? If we interpret it as an **unsigned** binary integer, what is the corresponding decimal value?

Solution : Signed: -11, Unsigned: 53

[3] b. Consider the hexadecimal value **1F**. What is the corresponding binary value? What is the corresponding decimal value?

Solution : Binary: 011111, Decimal: 31

[1] c. Write down the two binary integers from parts (a) and (b). Then add them together in binary. What is the resulting binary integer (assuming that we only have 6 bits to represent it)? Please write down your answer in this form A + B = C where A, B, and C are binary numbers. For example, this can be a possible answer: 001000 + 101010 = 111100.

Solution : 110101 + 011111 = 010100

[1] d. Consider your answer A + B = C from part (c). Suppose that we interpret all three binary integers A, B, and C as unsigned binary integers. Convert all three binary integers into decimal values and write down the same addition in decimal. For example, this can be a possible answer: 30 + 60 = 120.

Solution : 53 + 31 = 20

[2] e. Does your answer from part (d) make sense? Why or why not?

Solution : It does (or not, depending on how you define "making sense"), because $53 + 31 = 84$, and 84 is larger than the maximum value we can represent using unsigned 6 bit binary integers (which is 63).

[8] 4. Determine the validity of the following argument *using rules of inference and/or logical equivalences*. You must state what inference rule or logical equivalence you use at each step. Do not rewrite the premises; start your proof at step 6.

1. $p \oplus q$
2. $(s \to r) \to t$
3. $(t \to m) \lor u$
4. $r \lor \sim q$
5. $\sim u \land \sim m$

$\therefore\ p \land s$

Solution :

6.	$\sim u$	Specialization from (5).
7.	$\sim m$	Specialization from (5).
8.	$t \to m$	Elimination from (3) and (6).
9.	$\sim t$	Modus tollens from (7) and (8).
10.	$\sim(s \to r)$	Modus tollens from (2) and (9).
11.	$\sim(\sim s \vee r)$	Definition of $\to$ from (10).
12.	$\sim\sim s \wedge \sim r$	De Morgan's law from (11).
13.	$s \wedge \sim r$	Double negative law from (12).
14.	s	Specialization from (13).
15.	$\sim r$	Specialization from (13).
16.	$\sim q$	Elimination from (4) and (15).
17.	$(p \vee q) \wedge \sim(p \wedge q)$	Definition of $\oplus$ from (1).
18.	$p \vee q$	Specialization from (17).
19.	p	Elimination from (16) and (18).
20.	$p \wedge s$	Conjunction from (14) and (19).

So the argument is valid.

[3] 5. You were asked to prove that an argument such as the following is valid:

1. *premise* 1
2. *premise* 2
3. *premise* 3

$\therefore\ p \wedge s$

but instead of proving $p \wedge s$ you succeeded in proving $\sim(p \wedge s)$. A teaching assistant verified your proof and confirmed (correctly!) that you did not make a mistake. What are the two possible reasons for the perplexing conclusion you reached?

Solution : One possible reason is that the argument is invalid. Thus $p \wedge s$ must indeed be false whenever all three premises hold. The other possibility is that the premises contradict one another, in which case you can prove both $p \wedge s$ and $\sim(p \wedge s)$.

[6] 6. Consider the following definitions:

- P: the set of all people living around 200 BC.
- T: the set of tools/weapons available at that time.
- $G(x)$: person x is Greek
- $R(x)$: person x is Roman
- $W(x, y)$: person x wielded weapon y

- *Talked*(x, y): person x talked to person y
- *Fought*(x, y): person x fought person y

Translate each of the following English statements into predicate logic. For instance, "Alex wielded some weapon" would be translated as $\exists x \in T, W(\text{Alex}, x)$.

[3] a. A Greek wielding a bronze sword (an element of T) fought a Roman wielding an iron shield (another element of T).

Solution: $\exists x \in P\, \exists y \in P, G(x) \wedge R(y) \wedge W(x, \text{"bronze sword"}) \wedge W(y, \text{"iron shield"}) \wedge \textit{Fought}(x, y)$.

[3] b. People who used cudgels (an element of T) fought every Greek they talked to.

Solution: $\forall x \in P\, \forall y \in P, (W(x, \text{"cudgel"}) \wedge G(y) \wedge \textit{Talked}(x, y)) \rightarrow \textit{Fought}(x, y)$.

[3] 7. Using the same definitions in the previous question, translate the following predicate logic statements into English:

$$\forall x \in P, (G(x) \wedge \exists y \in P, R(y) \wedge \textit{Talked}(x, y)) \rightarrow (\exists z \in P, G(z) \wedge x \neq z \wedge \textit{Talked}(x, z))$$

Solution: Every Greek who talked to a Roman also talked to another Greek.

[3] 8. Why do we almost never translate an English sentence into a predicate logic statement of the form $\exists x \in D, P(x) \rightarrow Q(x)$?

Solution: It is because such a predicate logic statement would be true every time there is an element x of the domain for which $P(x)$ is false, which means it's not saying very much.

[9] 9. In this question, you will design a circuit that takes as input a 4-bit unsigned binary integer $x_3x_2x_1x_0$, and outputs its integer square root as a 2-bit unsigned binary integer y_1y_0. By "integer square root", we mean that any fractional part is discarded. For instance, the integer square root of 11 is 3 because $\sqrt{11} = 3.3166\ldots$

a. Write a proposition for the value of y_1. Any correct proposition will be worth at least 1.5/2. In order to get 2/2, you need to write one that is not too ugly. Hint: our solution is very short.

Solution: The bit y_1 will be 1 whenever the answer is 2 or 3. This happens when the input value is ≥ 4, which is when either x_3 or x_2 is 1. The proposition is therefore $x_3 \vee x_2$.

b. Write a proposition for the value of y_0. Any correct proposition will be worth at least 2.5/4. In order to get 4/4, you need to write one that is not too ugly.

Solution : Let us start by writing a truth table for y_0. As we are interpreting the four inputs as a single unsigned binary integer, we will use 0 and 1 instead of F and T in the table.

x_3	x_2	x_1	x_0	y_0
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Our first observation is that there is only case where x_3 is 1 and the output y_0 is 0: the case where x_2, x_1 and x_0 are all 0. Thus one condition we want to detect is $x_3 \wedge (x_2 \vee x_1 \vee x_0)$. The other cases are when $x_3 = x_2 = 0$, and at least one of x_1, x_2 is 1. This gives us the proposition $\sim x_3 \wedge \sim x_2 \wedge (x_1 \vee x_0)$. One answer is thus the proposition

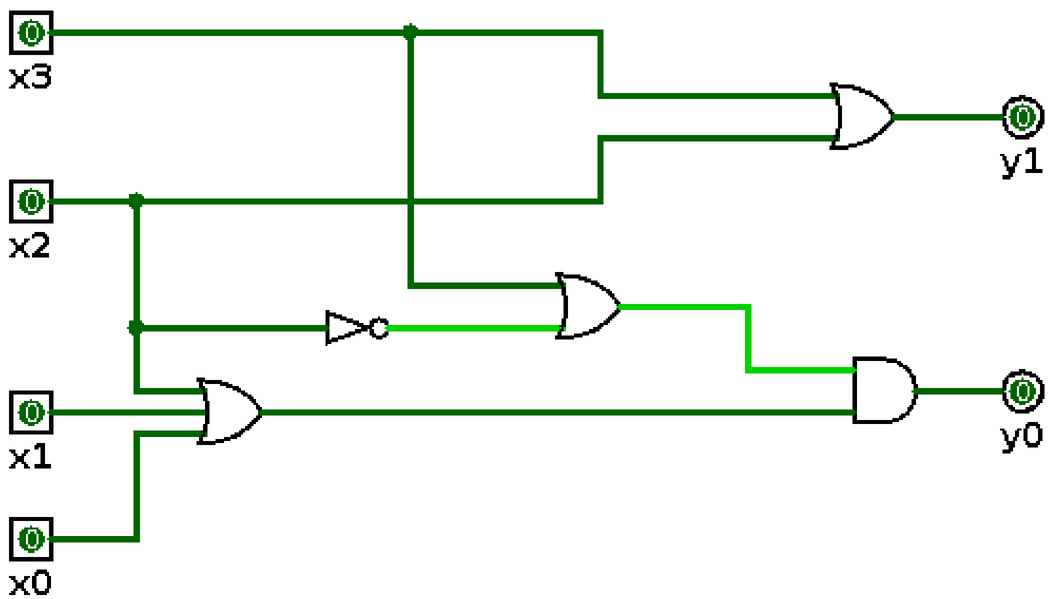
$$(x_3 \wedge (x_2 \vee x_1 \vee x_0)) \vee (\sim x_3 \wedge \sim x_2 \wedge (x_1 \vee x_0))$$

Here is another, shorter, answer: the cases where y_0 is 1 are those were $x_3 x_2$ is not the pair 01, except for when $x_2 x 1_x 0$ is 000. So we get

$$(x_3 \vee \sim x_2) \wedge (x_2 \vee x_1 \vee x_0)$$

c. Finally draw your circuit below.

Solution :


0
x3
0
x2
0
x1
0
x0
0
y1
0
y0