CPSC 121 Midterm 2
Friday November 13th, 2015

[10] 1. For each of the following statements:

- Rewrite the statement using predicate logic.
- Write down ALL of the proof strategies (from the list given below), that are appropriate to use to prove the statement.

Just use the LETTER for each strategy (do not write down the full name):

A. Constructive or non-constructive direct proof of existence.
B. Exhaustive proof.
C. Direct proof by generalizing from a generic particular (i,e, let x be any non particular (unspecified)...).
D. Proof by cases.
E. Contrapositive proof.
F. Antecedent assumption (called direct proof by Epp).
G. Proof by contradiction.

Note that you must **NOT** prove the statement. You just need to suggest all the proof strategies which can be used to prove it.

[2] a. For any three integers a, b and c, if a divides b and b divides c then a divides c. Use $\mathbf{Z}$ to represent the set of integers, and predicate *Divides*(x, y) for "x divides y".

Solution : The predicate logic statement is:

$$\forall a \in \mathbf{Z} \forall b \in \mathbf{Z} \forall c \in \mathbf{Z}, \textit{Divides}(a, b) \wedge \textit{Divides}(b, c) \rightarrow \textit{Divides}(a, c)$$

Possible proof strategies are C, F and G.

[2] b. For any positive real numbers x and y, $\frac{x+y}{2} \geq \sqrt{xy}$. Use $\mathbf{R}^+$ to represent the set of all positive real numbers.

Solution : The predicate logic statement is:

$$\forall x \in \mathbf{R}^+ \forall y \in \mathbf{R}^+, \frac{x+y}{2} \geq \sqrt{xy}$$

A possible proof strategy is G.

[2] c. There is a perfect square that is the sum of two perfect squares. Use $\mathbf{Z}$ to represent the set of integers, and predicate *Square*(n) for "n is a perfect square".

Solution : The predicate logic statement is:

$$\exists x \in \mathbf{Z} \exists y \in \mathbf{Z} \exists z \in \mathbf{Z}, \textit{Square}(x) \wedge \textit{Square}(y) \wedge \textit{Square(z)} \wedge x + y = z$$

A possible proof strategy is A.

[2] d. For any integers n and m, $n + m$ is odd if and only if $n - m$ is odd. Use predicates *Odd*(x) for "x is odd" and *Even*(x) for "x is even".

Solution : The predicate logic statement is:

$$\forall n \in \mathbf{Z} \forall m \in \mathbf{Z}, \textit{Odd}(n + m) \leftrightarrow \textit{Odd}(n - m)$$

Possible proof strategies are C, E and F.

[2] e. The product of a non-zero rational number x and an irrational number y is an irrational number. Use $\mathbf{R}$ for the domain of real numbers, and predicates *Rational*(x) for "x is rational" and *Irrational*(x) for "x is irrational".

Solution : The predicate logic statement is:

$$\forall x \in \mathbf{R} \forall y \in \mathbf{R}, \textit{Rational}(x) \wedge x \neq 0 \wedge \textit{Irrational}(y) \rightarrow \textit{Irrational}(xy)$$

A possible proof strategy is G.

[8] 2. Provide a formal proof (by using one formal rule of inference at a time and explicitly stating the rule that is used on the right side) for the following argument. You don't need to rewrite the premises in the proof. Just continue with step 4.

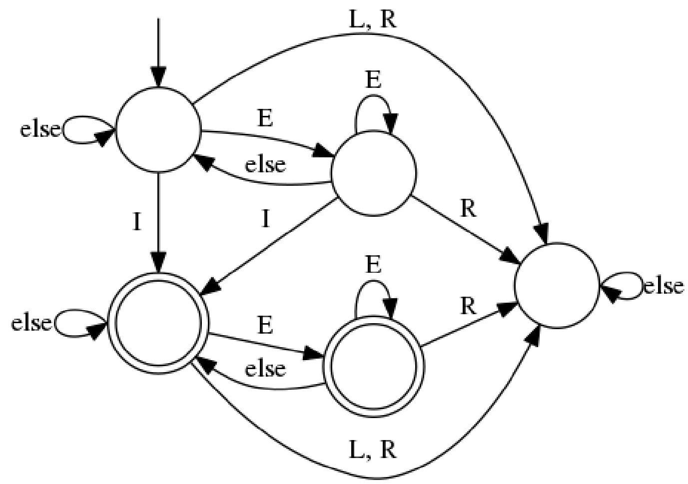
1. $\forall x \in D, \forall y \in D, P(x) \rightarrow (Q(y) \wedge R(x, y))$
2. $\forall z \in D, \sim Q(z) \vee R(z, z)$
3. $\exists u \in D, P(u)$

$\therefore\ \exists x \in D, R(x, x)$

Solution :

4.	$P(w)$	Existential instantiation from (3).
		w is a new unspecified element of D.
5.	$\forall y \in D, P(w) \rightarrow (Q(y) \wedge R(w, y))$	Universal instantiation from (1).
6.	$P(w) \rightarrow (Q(w) \wedge R(w, w)$	Universal instantiation from (5).
7.	$Q(w) \wedge R(w, w)$	Modus ponens from (4) and (6).
8.	$R(w, w)$	Specialization from (7).
9.	$\exists x \in D, R(x, x)$.	Existential generalization from (8).

[7] 3. Consider the following deterministic finite-state automaton, whose inputs are words written using uppercase letters (A through Z) only. The word "else" on an edge denotes every letter that does not appear on any other edge leaving from the same node as that edge (for instance, the word "else" on the edge that joins the top-left node to itself is the same as "A, . . . , D, F, G, H, J, K, M, . . . , Q, S, . . . , Z").



[4] a. Which of the following words will this finite-state automaton accept (circle one of Yes/No for each word)?

Solution :

- **INDOLENT** Yes $\boxed{\text{No}}$
- **RABBITS** Yes $\boxed{\text{No}}$
- **LIKE** Yes $\boxed{\text{No}}$
- **WATCHING** $\boxed{\text{Yes}}$ No
- **SIR** Yes $\boxed{\text{No}}$
- **ELVIS** $\boxed{\text{Yes}}$ No
- **ON** Yes $\boxed{\text{No}}$
- **TELEVISION** $\boxed{\text{Yes}}$ No

[3] b. Describe as simply as you can the set of words that this finite-state automaton accepts.

Solution : It accepts words that contain at least one I, do not contain any R, and where every L must be immediately preceded by an E.

[11] 4. For each of the two following theorems, first translate the theorem into predicate logic, and then prove it using a proof technique of your choice.

a. It is possible to find positive integers x, y and z such that $x^4 + y^3 = z^2$.

Solution : Translated into predicate logic, this is $\exists x \in \mathbf{Z}^+, \exists y \in \mathbf{Z}^+, \exists z \in \mathbf{Z}^+, x^4 + y^3 = z^2$.

To prove this statement, we only need to choose three integers x, y and z and to verify that $x^4 + y^3 = z^2$. So choose $x = 1$, $y = 2$ and $z = 3$. It is easily verified that $1^4 + 2^3 = 1 + 8 = 9 = 3^2$. QED

b. If a, b and c are positive integers, and a divides both $b + c$ and bc, then a divides $b^2 + c^2$.

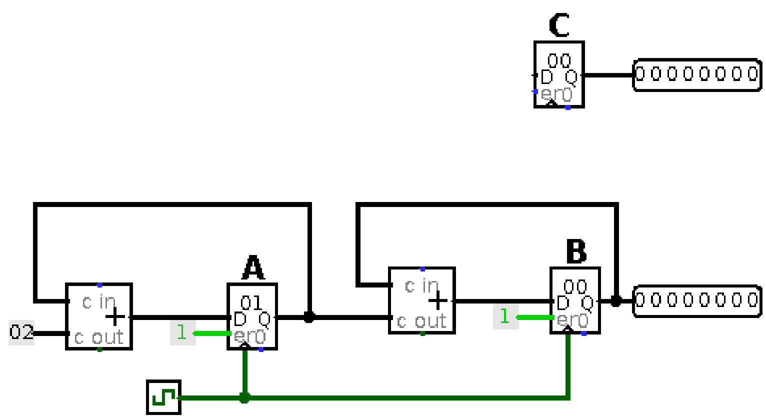
Solution : Translated into predicate logic, this is $\forall a \in \mathbf{Z}^+, \forall b \in \mathbf{Z}^+, \forall c \in \mathbf{Z}^+, a \mid b + c \wedge a \mid bc \rightarrow a \mid b^2 + c^2$ (recall that $a \mid b$ means "a divides b").

To prove this, consider three unspecified positive integers a, b, c. Assume that a divides $b + c$ and that a divides bc. Hence $b + c = ax$ for some integer x and $bc = aqy$ for some integer y. Then

$$b^2 + c^2 = (b + c)^2 - 2bc = (ax)^2 - 2ay = ax^2 = 2ay = a(x^2 - 2y).$$

Since x and y are both integers, so is $x^2 - 2y$, which means that a divides $b^2 + c^2$. QED

[7] 5. Consider the following sequential circuit:



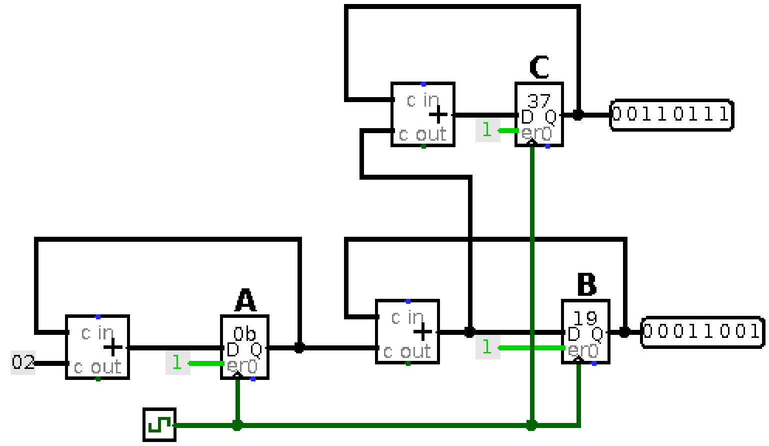
Please observe that the registers labeled A, B and C initially contain the values 1, 0 and 0 respectively.

[3] a. What values will registers A and B contain five clock cycles later?

Solution : At the end of each of these five cycles, register A will contain the values 3, 5, 7, 9 and 11, while register B will contain 1, 4, 9, 16 and 25.

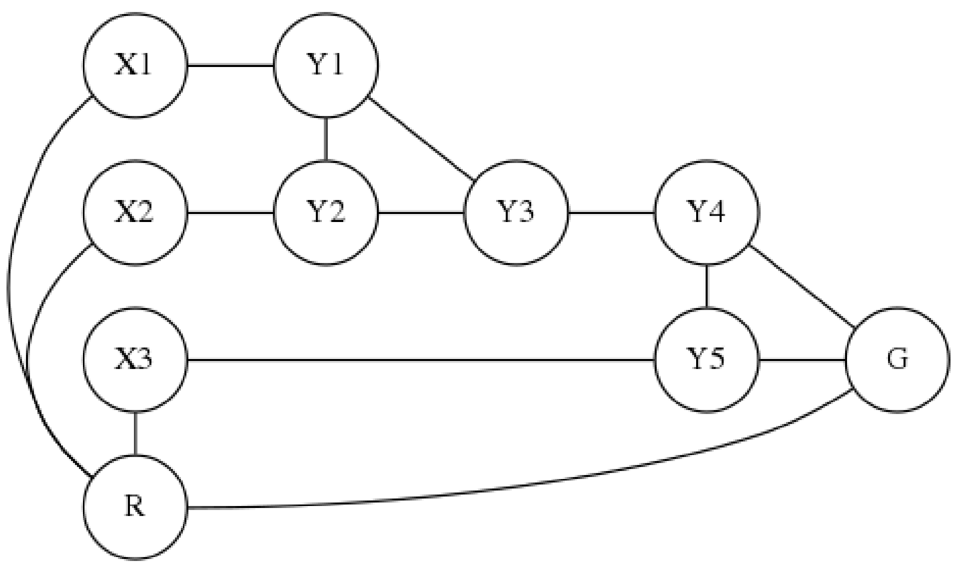
[4] b. Let $f(n)$ denote the value of register B after n clock cycles. Complete the circuit so that, after n clock cycles, register C contains the value $f(1) + f(2) + \cdots + f(n)$, that is, the sum of the values that register B held during the n clock cycles. For instance, if register B had values 1, 8 and 3 at the end of clock cycles 1, 2 and 3, then C would take on the values 1, 9 and 12.

Solution :



[7] 6. Suppose that we want to color the nodes (circles) in the following figure according to four rules:

- Every node must be colored using one of the colors Red, Green and Blue.
- Two nodes that are joined by a line **must** have different colors.
- The node labeled "R" must be colored Red.
- The node labeled "G" must be colored Green.



Using a proof by contradiction, prove that every coloring that follows these four rules must use the color Green for at least one of the nodes labeled X1, X2, X3. Hint: if three nodes are all joined by a line to each other, then they will use three different colors.

Solution : We use a proof by contradiction (as stated in the question!). Suppose that none of X1, X2, X3 are colored green. Since all of them are joined to R, they can not be colored red, and hence they are blue. Now Y1, Y2 and Y3 are all joined to each other, so they must use three different colors. Neither Y1 nor Y2 can be blue (they're both connected to a blue node already), so Y3 must be blue. Using the same reasoning for Y4, Y5 and G, we see that G must also be blue. But that contradicts the rule that says G must be green. QED