CPSC 121 Midterm 1
Friday February 5th, 2016

[3] 1. Is the following statement a tautology, a contradiction or neither? Justify your answer using a truth table:

$$(p \leftrightarrow q) \rightarrow (p \oplus \sim q)$$

Solution : This statement is a tautology:

p	q	$p \leftrightarrow q$	$\sim q$	$p \oplus \sim q$	$(p \leftrightarrow q) \rightarrow (p \oplus \sim q)$
F	F	T	T	T	T
F	T	F	F	F	T
T	F	F	T	F	T
T	T	T	F	T	T

[6] 2. Using a sequence of logical equivalences, prove that

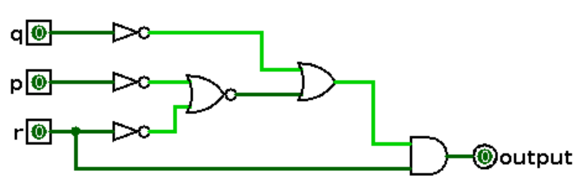
$$(\sim p \wedge \sim q) \vee \sim(\sim(r \wedge q) \rightarrow p) \equiv \sim p \wedge (\sim q \vee \sim r)$$

Write the name of the law you applied at each step.

Solution : The left-hand side (LHS) is $(\sim p \wedge \sim q) \vee \sim(\sim(r \wedge q) \rightarrow p)$. Hence

LHS	$\equiv$	$(\sim p \wedge \sim q) \vee \sim(\sim\sim(r \wedge q) \vee p)$	by the definition of implication
	$\equiv$	$(\sim p \wedge \sim q) \vee \sim((r \wedge q) \vee p)$	by the double negative law
	$\equiv$	$(\sim p \wedge \sim q) \vee (\sim(r \wedge q) \wedge \sim p)$	by the de Morgan's law
	$\equiv$	$(\sim p \wedge \sim q) \vee (\sim p \sim(r \wedge q))$	by the commutative law
	$\equiv$	$\sim p \wedge (\sim q \vee \sim(r \wedge q))$	by the distributive law
	$\equiv$	$\sim p \wedge (\sim q \vee (\sim r \vee \sim q))$	by the de Morgan's law
	$\equiv$	$\sim p \wedge ((\sim q \vee \sim q) \vee \sim r)$	by the commutative and associative laws
	$\equiv$	$\sim p \wedge (\sim q \vee \sim r)$	by the idempotent law

[9] 3. Consider the following digital circuit:



[3] a. Write a propositional logic expression which is the direct translation of this circuit to propositional logic. Do not simplify your expression.

Solution : $(\sim(\sim p \vee \sim r) \vee \sim q) \wedge r$

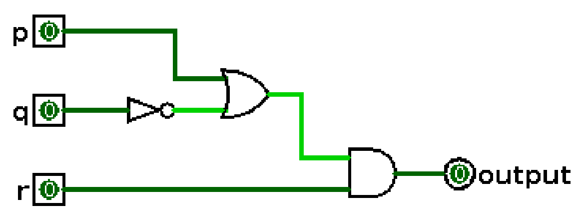
[4] b. Simplify your answer from part (a) using a sequence of logical equivalences. Write the name of the law you applied at each step.

Solution :

$$\begin{array}{rcll}
(\sim(\sim p \vee \sim r) \vee \sim q) \wedge r & \equiv & ((\sim\sim p \wedge \sim\sim r) \vee \sim q) \wedge r & \text{by the de Morgan's law} \\
& \equiv & ((p \wedge r) \vee \sim q) \wedge r & \text{by the double negative law} \\
& \equiv & ((p \vee \sim q) \wedge (r \vee \sim q)) \wedge r & \text{by the distributive law} \\
& \equiv & (p \vee \sim q) \wedge ((r \vee \sim q) \wedge r) & \text{by the associative law} \\
& \equiv & (p \vee \sim q) \wedge r & \text{by the absorption law}
\end{array}$$

[2] c. Draw the simplified circuit corresponding to your answer for part (b). Hint: you should only need 3 gates.

Solution :



[11] 4. Assume that we use 6 bits to represent signed binary integers (using two's complement).

[3] a. What is the 6-bit binary representation of the signed decimal value -15? What is the corresponding hexadecimal representation using 2 digits?

Solution : The binary representation of 15 is 001111; hence the binary representation of -15 is $110000 + 1 = 110001$. In hexadecimal, this is 31.

[3] b. What is the 6-bit binary representation of the hexadecimal value 2E? What is the corresponding signed decimal value?

Solution : The binary representation is 101110. This number is negative, and its inverse is $010001 + 1 = 010010 = 2^4 + 2 = 18$. Hence 2E is -18.

[1] c. With 6 bits, what is the smallest (most negative) signed binary integer that you can represent?

Solution : $-2^5 = -32$

[2] d. What is the sum of the two 6-bit signed binary integers from parts (a) and (b)? What is the corresponding decimal value?

Solution : $110001 + 101110 = 011111$, which is $16 + 8 + 4 + 2 + 1 = 31$.

[2] e. Why is your decimal answer from part (d) positive?

Solution : It is positive because $-15 + -18 = -33$, and -33 is outside the range of values that can be represented using 6-bit signed integers.

[12] 5. Three mafiosi (Don Cortizone, Don Maledictione and Don Sanconvictione) went to the horse races. Inspector Montalbano, who is watching them, learns that

- If Don Sanconvictione bet on Gallant Fox in the third race, then Don Maledictione paid Secretariat's jockey to lose.
- If Don Cortizone provided EPO to Whirlaway, then Don Sanconvictione and Don Maledictione bet on Count Fleet in the seventh race.
- Either Don Maledictione did not win any of his bets, or Don Sanconvictione was responsible for War Admiral's sickness (or both).
- If Don Sanconvictione and Don Maledictione bet on Count Fleet in the seventh race and Don Sanconvictione was responsible for War Admiral's sickness, then Don Maledictione did not pay Secretariat's jockey to lose.

After the races were over, Detective Catarella brought him proof that not only Don Cortizone provided EPO to Whirlaway, but also that Don Maledictione won all five of his bets.

[3] (a) Rewrite each of these statements using propositional logic. Make sure to define the propositions you are using by underlining part of the sentence above (for instance "Don Sanconvictione bet on Gallant Fox in the third race") and putting the letter (for instance, "g") next to it.

Solution : Let

gf:	Don Sanconvictione bet on Gallant Fox in the third race.
s:	Don Maledictione paid Secretariat's jockey to lose.
w:	Don Cortizone provided EPO to Whirlaway.
cf:	Don Sanconvictione and Don Maledictione bet on Count Fleet in the seventh race.
b:	Don Maledictione won some of his bets.
wa:	Don Sanconvictione was responsible for War Admiral's sickness.

Then the statements can be written as:

1. $gf \rightarrow s$
2. $w \rightarrow cf$
3. $\sim b \vee wa$
4. $(cf \wedge wa) \rightarrow \sim s$
5. $w \wedge b$

[9] (b) Using your answer from part (a), known logical equivalences and the rules of inference, prove that Don Sanconvictione did not bet on Gallant Fox in the third race. Write the name of the equivalence law or the inference rule you applied at each step.

Solution : We are trying to prove $\sim gf$. Here is one possible proof (there may be others).

6.	w	Specialization from (5).
7.	cf	Modus ponens from (2) and (6).
8.	b	Specialization from (5).
9.	wa	Elimination from (3) and (8).
10.	$cf \wedge wa$	Addition from (7) and (9).
11.	$\sim s$	Modus ponens from (4) and (10).
12.	$\sim gf$	Modus tollens from (1) and (11).

[6] 6. Consider the following definitions:

- C: the set of all colors.
- S: the set of all shoppers.
- I: the set of all items (dress, shoes, pants, hats, gloves, shirts, etc).
- $\mathit{Knows}(x, y)$: shopper x knows shopper y.
- $\mathit{Bought}(x, y, z)$: shopper x bought item y in color z.

translate each of the following English statements into predicate logic. For instance, "Mrs. Nelly bought a colored skirt" would be translated as $\exists x \in C, \mathit{Bought}(\text{Mrs. Nelly}, \text{skirt}, x)$.

[3] a. A shopper who knows Cédric bought a blue shirt.

Solution : $\exists x \in S, \mathit{Knows}(x, \text{Cédric}) \wedge \mathit{Bought}(x, \text{shirt}, \text{blue})$.

[3] b. Every shopper who bought green gloves knows a shopper who bought a red hat.

Solution : $\forall x \in S, \mathit{Bought}(x, \text{gloves}, \text{green}) \rightarrow \exists y \in S, \mathit{Knows}(x, y) \wedge \mathit{Bought}(y, \text{hat}, \text{red})$.

[6] 7. Using the same definitions as for the previous question, translate each of the following predicate logic statements into English.

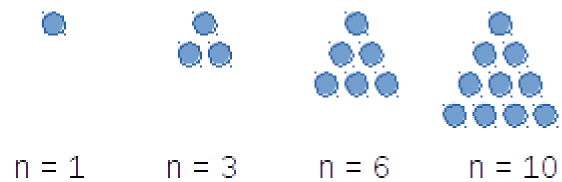
[3] a. $\forall x \in I, \forall y \in C, \mathit{Bought}(\text{Nicolas}, x, y) \rightarrow \mathit{Bought}(\text{Chen}, x, y)$.

Solution : Chen bought every item that Nicolas bought, in the same color.

[3] b. $\exists x \in S, \forall y \in S, \textit{Bought}(y, \text{dress}, \text{pink}) \to \textit{Knows}(x, y)$.

Solution : Someone knows every shopper who bought a pink dress.

[7] 8. A triangular number (or triangle number) counts the objects that can form an equilateral triangle, as in the diagram below. The n^{th} triangular number is the sum of the first n integers, as shown in the following figure illustrating the first four triangular numbers (what is the fifth one?):



Design a circuit that takes a 4-bit unsigned binary integer $x_3x_2x_1x_0$ as input, and outputs True if this integer is a triangular number, and False otherwise. You may assume that 0 is not a triangular number.

Solution : The only four-bit triangular numbers $x_3x_2x_1x_0$ are 1 (0001), 3 (0011), 6 (0110), 10 (1010) and 15 (1111). We thus need a proposition (and a circuit) that matches exactly these five bit patterns, and no other one. Here are a couple of observation about these bit patterns:

- The first two have $x_3 = x_2 = 0$ and $x_0 = 1$.
- The next two have exactly one of x_3, x_2 equal to 1, and end with 10 ($x_1 = 1$ and $x_2 = 0$).

So the proposition $\sim x_3 \wedge \sim x_2 \wedge x_0$ matches the first two patterns, the proposition $(x_3 \oplus x_2) \wedge x_1 \wedge \sim x_0$ matches the next two, and $x_3 \wedge x_2 \wedge x_1 \wedge x_0$ matches the last one. This leads us to the following circuit, where each input to the final OR gate corresponds to one of the three propositions: