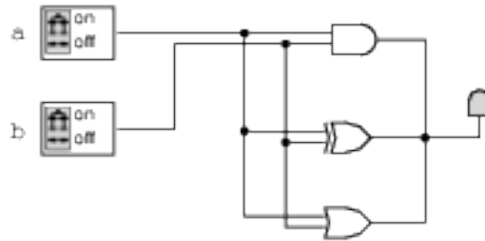


CPSC 121 Sample Final Examination
December 2013

[6] 1. Short answers

[3] a. What is wrong with the following circuit?



Solution: You can not connect the outputs of two or more gates together directly; what will happen if one of the gates tries to drive it high while the others try to drive it low?

[3] b. Explain briefly the difference between the purpose of a half-adder and that of a full-adder.

Solution: A half-adder computes the sum of two bits. A full-adder adds three bits (two bits and a carry from a previous position).

[6] 2. Let $A = \{1, 2, 3, \dots, 299, 300\}$.

[3] a. Give an example of a function f from \mathbf{Z}^+ into A that is not onto (surjective).

Solution: The function f defined by $f(x) = 1$ is not onto (there is no positive integer x such that $f(x) = 2$).

[3] b. Consider the function $f : \{1, 2, \dots, 18\} \rightarrow A$ defined by $f(x) = x^2 - 3x + 3$. Is f one-to-one (injective)? Explain why or why not.

Solution: No, because $f(1) = f(2) = 1$.

[9] 3. Number representation

[3] (a) What unsigned integer is represented by the 9-bit value 100111011?

Solution: $256 + 32 + 16 + 8 + 2 + 1 = 315$

[3] (b) What signed integer is represented by the 9-bit value 100111011?

Solution: $-256 + 32 + 16 + 8 + 2 + 1 = -197$

[3] (c) A professional programmer wrote a program that takes a floating point number x , sets its value to 0, and then adds 0.1 to x repeatedly until it becomes equal to 10, printing each value of x as it goes. Why does the program produce the output

```
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.70000005 0.8000001 0.9000001
1.0000001 1.1000001 1.2000002
```

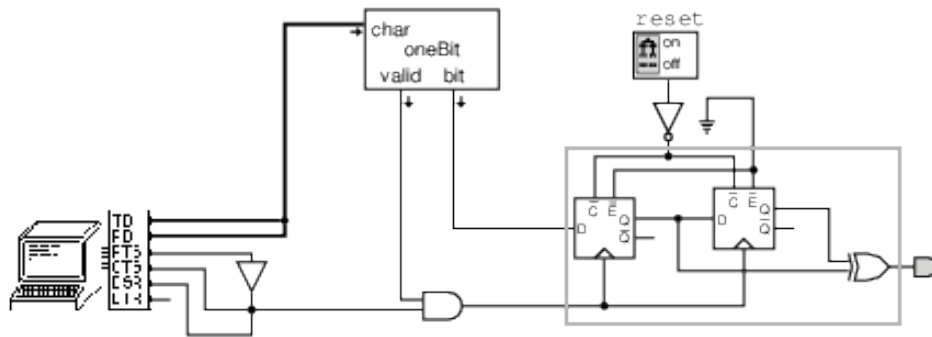
(and so on) instead of

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

?

Solution: The value 0.1 can not be represented exactly as a (binary) floating point value. Hence the value being added is not actually 0.1, but a number that's just a little bit larger than 0.1. This also means that the sum will never be exactly equal to 10 (and so the program will keep going, and going, and going, and ...)

[12] 4. Consider the following circuit:



The terminal and the `oneBit` subcircuit provide bits one at a time to the remainder of the circuit, and the clock attached to the D flip-flops ticks each time a bit is available.

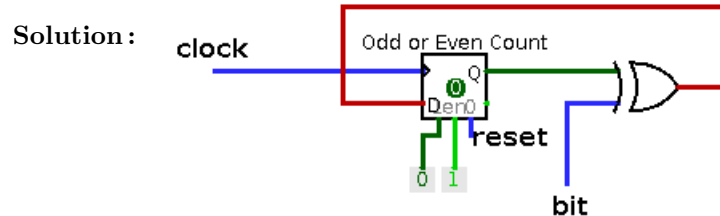
[4] a. Suppose that the user first flips the `reset` switch on, then off, and finally types in the digits 0, 0, 1, 1, 0, 1, 1, 0 in this order. What will be the state of the LED (on/off) after each of the digits is typed (you need to provide 8 answers).

Solution: The LED will be off, off, on, off, on, on, off and on respectively.

[4] b. Explain *in English* the purpose of this circuit.

Solution: The output is true (on) if the last two bits entered were different, and false (off) if they were the same.

[4] c. Show how to modify the part of the circuit that is inside the gray box so the LED would be **on** if an odd number of 1 digits had been typed, and **off** if an even number of 1 digits had been typed. Hint: you will only need one flip-flop, and a few other gates.



[7] 5. Determine the validity of the following argument. Justify your answer **fully**.

1. $(\sim p \vee q) \rightarrow r$
2. $r \rightarrow (s \vee t)$
3. $\sim s \wedge \sim u$
4. $\sim u \rightarrow \sim t$

$\therefore p$

Solution :

- | | | |
|-----|----------------------------|---------------------------------|
| 5. | $\sim u$ | specialization from (3) |
| 6. | $\sim t$ | modus ponens from (3) and (5) |
| 7. | $\sim s$ | specialization from (3) |
| 8. | $\sim s \wedge \sim t$ | conjunction from (6) and (7) |
| 9. | $\sim(s \vee t)$ | DeMorgan's law from (8) |
| 10. | $\sim r$ | modus tollens from (2) and (9) |
| 11. | $\sim(\sim p \vee q)$ | modus tollens from (1) and (10) |
| 12. | $\sim\sim p \wedge \sim q$ | deMorgan's law from (11) |
| 13. | $\sim\sim p$ | specialization from (12) |
| 14. | p | double negative law from (13) |

Therefore the argument is valid.

[9] 6. A *perfect number* is a positive integer that is equal to the sum of its divisors (except itself). For instance, 28 is perfect since $28 = 1 + 2 + 4 + 7 + 14$. In this question, we will look at *almost-perfect numbers*: numbers that are one more than the sum of their divisors (except themselves). For instance, 16 is almost perfect since $1 + 2 + 4 + 8 = 15$. Consider the following statement:

Given any positive integer n , we can always find an almost-perfect number that is larger than n .

[3] a. Translate this statement into first-order logic (that is, using predicates and quantifiers). You may assume the existence of a predicate `isAlmostPerfect`.

Solution : $\forall n \in \mathbf{Z}^+, \exists x \in \mathbf{Z}^+, x > n \wedge \text{isAlmostPerfect}(x)$

[6] b. Prove the statement. Hint: look carefully at the example I gave, and see if you can make it more general.

Solution : Consider an unspecified positive integer n . There is a power of 2 (say $x = 2^n$) that is larger than n . The divisors of x are $2^0, 2^1, \dots, 2^{n-1}, 2^n$. Hence the sum of the divisors (other than x) is

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

which means that x is almost perfect.

[11] 7. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and f be a function from \mathcal{U} into \mathcal{U} . Finally, let $F : \mathcal{P}(\mathcal{U}) \rightarrow \mathcal{P}(\mathcal{U})$ be the function defined by

$$F(X) = \{f(x) \mid x \in X\}$$

For instance, if $f(x) = 13 - x$, and $X = \{1, 3, 4, 7\}$, then $F(X) = \{f(1), f(3), f(4), f(7)\} = \{12, 10, 9, 6\}$.

[7] a. Prove that for every two subsets A, B of \mathcal{U} , we have $F(A \cap B) \subseteq F(A) \cap F(B)$.

Solution: Consider two unspecified subsets A, B of \mathcal{U} , and choose an arbitrary element x of $F(A \cap B)$. We need to prove that $x \in F(A) \cap F(B)$.

Since $x \in F(A \cap B)$, there exists an element y of $A \cap B$ such that $x = f(y)$. Because $y \in A \cap B$, we know that $y \in A$ and hence $x = f(y) \in F(A)$. Similarly $y \in B$ and so $x = f(y) \in F(B)$. This implies that $x \in F(A) \cap F(B)$.

[4] b. Give an example that shows that $F(A - B)$ is not necessarily a subset of $F(A) - F(B)$.

Solution: Let $A = \{1, 2\}$, $B = \{1\}$ and $f(x) = 1$ for every $x \in \mathcal{U}$. Then $F(A - B) = F(\{2\}) = \{1\}$, but $F(A) - F(B) = \{1\} - \{1\} = \emptyset$.

[8] 8. Prove that for every positive integer n ,

$$\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$$

Solution: We prove the theorem by induction on n . When $n = 1$,

$$\sum_{i=1}^n i(i+1) = 1 \times 2 = 2$$

and

$$\frac{n(n+1)(n+2)}{3} = \frac{1 \times 2 \times 3}{3} = 2$$

. Hence the theorem holds for the base case $n = 1$. Now let us prove the induction step. Consider an unspecified positive integer $n \geq 1$. Suppose that the theorem holds for n , that is suppose that

$$\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$$

. We need to prove that the theorem holds for $n + 1$. Indeed

$$\begin{aligned} \sum_{i=1}^{n+1} i(i+1) &= \left(\sum_{i=1}^n i(i+1) \right) + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) && \text{by the induction hypothesis} \\ &= (n+1)(n+2) \left(\frac{n}{3} + 1 \right) \\ &= (n+1)(n+2) \frac{n+3}{3} \\ &= \frac{(n+1)(n+2)(n+3)}{3} \end{aligned}$$

- [8] 9. The inhabitants of the country of Pirloutland use only three kinds of coins, worth 1 schlip, 5 schlips and 10 schlips respectively. Johan, one of the knights, makes the following observation:

For every amount $n \geq 10$ schlips, the way to pay for n schlips that uses the fewest coins always contains a 10 schlip coin.

Prove Johan's observation using *an indirect proof*. Hint: think of substituting some coin(s) for other coin(s).

Solution: We start by assuming that the conclusion is false, that is, that there exists an amount $n \geq 10$ schlips for which the way to pay this amount with the fewest coin does **not** use a 10 schlip coin. There are three cases to consider:

- If the smallest collection of coins that adds up to n schlips contains two or more 5 schlip coins, then we can replace two of the 5 schlip coins by a single 10 schlip coin. This gives us a way to pay for n schlips using fewer coins than the solution that uses the fewest coins (a contradiction).
- If the smallest collection of coins that adds up to n schlips contains exactly one 5 schlip coins, then it must also contain at least five 1 schlip coins. We can replace the 5 schlip coin and five of the 1 schlip coins by a single 10 schlip coin. This gives us a way to pay for n schlips using fewer coins than the solution that uses the fewest coins (a contradiction).
- If the smallest collection of coins that adds up to n schlips does not contain any 5 schlip coin, then it must also contain at least ten 1 schlip coins. We can replace ten of the 1 schlip coins by a single 10 schlip coin. This gives us a way to pay for n schlips using fewer coins than the solution that uses the fewest coins (a contradiction).

We found a contradiction no matter how our solution without 10 schlip coins was constructed, which means our assumption that such a solution exists was wrong.

- [12] 10. Mathematical Induction

- [3] a. Explain briefly the two forms of mathematical induction discussed in class.

Solution: In one form, we have a single base case, and for the induction step we assume that the theorem is true for an unspecified positive integer n , and then prove that it must also hold for $n + 1$. In the second form, we have one (or more) base cases, and for the induction step we assume that the theorem is true for **every** positive integer from 1 to an unspecified positive integer n , and then prove that it must also hold for $n + 1$.

- [9] b. Recall that $\lfloor x \rfloor$ is the largest integer that is $\leq x$, that $\lceil x \rceil$ is the smallest integer that is $\geq x$, and that $\forall x \in \mathbf{R}^+$, $\lfloor x/2 \rfloor + \lceil x/2 \rceil = x$.

Let p_1, p_2, p_3, \dots be a sequence of integers defined by

$$p_1 = 0, p_2 = 1 \quad \text{and} \\ p_n = p_{\lfloor n/2 \rfloor} + p_{\lceil n/2 \rceil} + 1 \quad \text{for all } n \in \mathbf{Z}^+ \text{ where } n \geq 3.$$

For example, $p_3 = p_{\lfloor 3/2 \rfloor} + p_{\lceil 3/2 \rceil} + 1 = p_1 + p_2 + 1 = 0 + 1 + 1 = 2$

Prove that for all nonnegative integers $n \geq 1$, $p_n = n - 1$. You must state the induction hypothesis clearly.

Solution: First we need to prove the base cases: $p_1 = 0$ which is indeed $1 - 1$ and $p_2 = 1$ which is $2 - 1$. Now consider an unspecified positive integer $n \geq 3$, and suppose that $p_k = k - 1$ for every k in the interval $1 \leq k < n$.

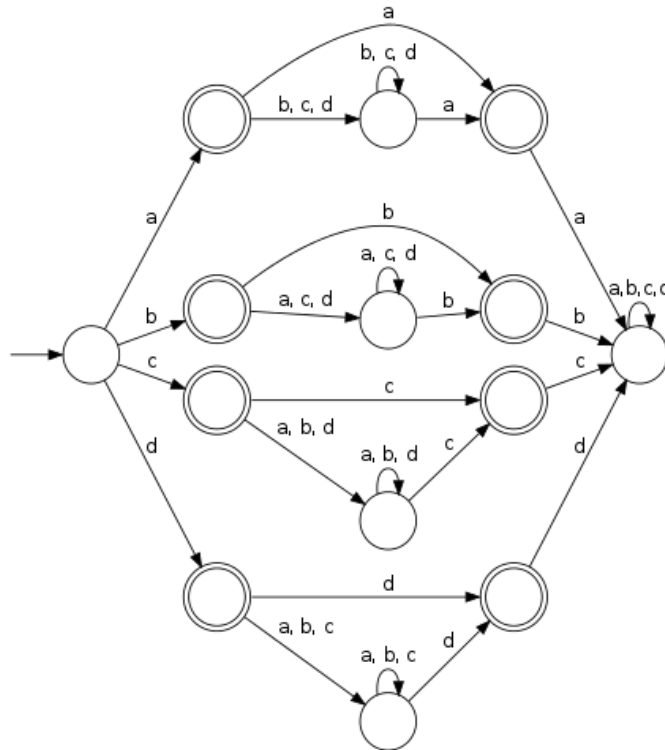
$$\begin{aligned}
 p_n &= p_{\lfloor n/2 \rfloor} + p_{\lceil n/2 \rceil} + 1 \\
 &= (\lfloor n/2 \rfloor - 1) + (\lceil n/2 \rceil - 1) + 1 && \text{by the induction hypothesis} \\
 &= (\lfloor n/2 \rfloor + \lceil n/2 \rceil) - 1 - 1 + 1 \\
 &= n - 1
 \end{aligned}$$

as required. This completes the induction step and the proof of the theorem.

[10] 11. Consider the set L of all strings over the alphabet $\{a, b, c, d\}$ that start and end with the same letter, and do not contain this letter in any other position. For instance, the strings a , bb and $cabbac$ belong to L , but the empty string and the strings $accb$ and $cabcac$ do not.

[7] a. Design a DFA that accepts every string in L , and reject every other string.

Solution:



[3] b. Write a regular expression that matches exactly the strings that belong to L .

Solution: $\hat{(a|b|c|d|a[bcd]^*a|b[acd]^*b|c[abd]^*c|d[abc]^*d)}\$$