CPSC 121 Midterm 2
Tuesday November 12th, 2013

[9] 1. Short Answers

[2] a. What kind of proof is the following (note: part of the proof is missing and has been replaced by ...) ?

> Consider an unspecified positive integer a. Suppose that the cube root of a is a rational number, but not an integer. This means that ... And therefore the cube root of a is irrational.

Solution : This is a proof by contradiction: we start by making an assumption (the cube root of a is a rational number, but not an integer) and finish by concluding the opposite.

[3] b. Consider the following statement:

$$\forall x \in \mathbf{Z}, \exists y \in \mathbf{Z}, Q(x, y)$$

Can a proof that starts with "Choose y = 3." be valid? Explain why or why not.

Solution : Yes: it is possible (but not likely) that the value $y = 3$ will work for every possible integer x. That is, maybe $Q(x, 3)$ is true for every $x \in \mathbf{Z}$. We could of course have chosen a different value of y for each x (and in fact in most cases we would need to do that), but it's not required.

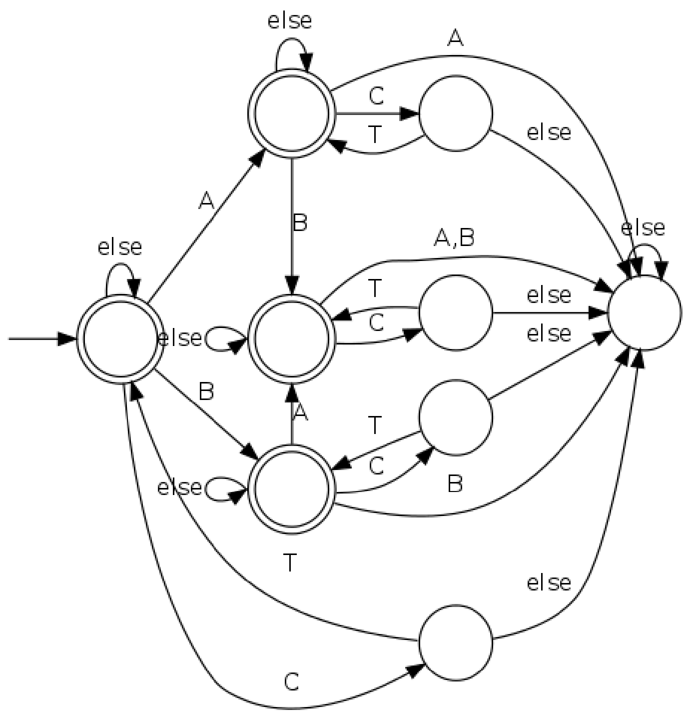
[4] c. You are asked to write a direct proof to show that the statement $\exists x \in D, \exists y \in D, P(x, y)$ is **false**. What would your proof look like? Write as much of the proof as you can, given that we did not tell you what $P(x, y)$ is.

Solution : We want to prove that this statement is false, or equivalently that its negation

$$\forall x \in D, \forall y \in D,\ P(x, y)$$

is true. Hence our proof would start with "Consider unspecified elements x, y of the domain D (where x may or may not equal y)". We would then try to show that $P(x, y)$ is false.

[7] 2. Consider the following deterministic finite-state automaton, whose inputs are words written using uppercase letters (A through Z) only. The word "else" on an edge denotes every letter that does not appear on any other edge leaving from the same node as that edge (for instance, the word "else" on the edge that joins the leftmost node to itself is the same as "D, E, F, ..., Y, Z").



[4] a. Which of the following words will this finite-state automaton accept (circle one of Yes/No for each word)?

Solution :

- **ACTIVE** — [Yes] No
- **DICTATION** — [Yes] No
- **DICTATORIAL** — Yes [No]
- **FOUGHT** — [Yes] No
- **LUMBER** — [Yes] No
- **PEBBLE** — Yes [No]
- **RABIES** — [Yes] No
- **SCHNITZEL** — Yes [No]

[3] b. Describe as simply as you can the set of words that this finite-state automaton accepts.

Solution : It accepts words that contain at most one A, at most one B, and where every C is immediately followed by a T.

[12] 3. For each of the following inferences, first state if it's valid or invalid, and then justify your answer briefly. Assume that a is an element of domain D.

[3] a.

$$\begin{array}{l} \exists x \in D, P(x) \to Q(x) \\ \sim Q(a) \\ \hline \therefore \sim P(a) \end{array}$$

Solution: This inference is invalid. Simply because the implication $P(x) \to Q(x)$ is true for one or more elements of the domain D, it does not mean that it is true for a (it could, for instance, be true for every element except a!). Since the implication $P(a) \to Q(a)$ may not be true, knowing $\sim Q(a)$ does not give us any information about $P(a)$.

[3] b.

$$\begin{array}{l} \forall x \in D, P(x) \to Q(x) \\ \sim Q(a) \\ \hline \therefore \sim P(a) \end{array}$$

Solution: This inference is valid: we can first use universal instantiation on the first premise to deduce that $P(a) \to Q(a)$, and then since we know $\sim Q(a)$ it follows from modus tollens that $\sim P(a)$.

[3] c.

$$\begin{array}{l} \exists y \in D(\exists x \in D, P(x, y)) \to Q(y) \\ \forall z \in D, P(z, a) \\ \hline \therefore\ Q(a) \end{array}$$

Solution: This inference is invalid: suppose we have $D = \{0, 1, \ldots, 10\}$, with $Q(y) : y = 0$ and $P(x, y) : x \leq y$ and $a = 10$.

- The first premise is valid: choose $y = 0$. Then $Q(y)$ holds, which means that any implication with $Q(y)$ as its right-hand side will be true.
- The second premise is valid: $z \leq a$ for every element of the domain (because a is the largest element of the domain).

However the conclusion $10 = 0$ is clearly wrong.

[3] d.

$$\begin{array}{l} \forall x \in D, \sim \forall y \in D, P(x, y) \to Q(x, y) \\ \hline \therefore\ \forall x \in D, \exists y \in D, P(x, y) \wedge \sim Q(x, y) \end{array}$$

Solution: This inference is valid: we simply applied the generalized DeMorgan's law to the premise to get

$$\forall x \in D, \exists y \in D, \sim (P(x, y) \to Q(x, y)),$$

used the definition of $\rightarrow$ to get

$$\forall x \in D, \exists y \in D, \sim(\sim P(x,y) \vee Q(x,y)),$$

and finally applied the DeMorgan's and double negative laws to obtain the desired conclusion.

[10] 4. For each of the two following theorems, first translate the theorem into predicate logic, and then prove it using a proof technique of your choice.

a. It is possible to find a pair of integers whose sum and difference are both perfect squares.

Solution : Translated into predicate logic, this is $\exists a \in \mathbf{Z}, \exists b \in \mathbf{Z}, \text{PerfectSquare}(a+b) \wedge \text{PerfectSquare}(a-b)$.

To prove this statement, we only need to choose two integers a and b and to verify that $a+b$ and $a-b$ are perfect squares. There were many possible pairs: in fact given any two perfect squares x, y that are either both even or both odd, and assuming without loss of generality that $x \leq y$, the two integers $a = (y+x)/2$ and $b = (y-x)/2$ would work (the sum is y, the difference is x).

This aside, all you needed to write was something like: "Choose $a = b = 0$. Then $a+b = 0 = 0^2$ and $a-b = 0 = 0^2$. Hence both $a+b$ and $a-b$ are perfect squares."

b. If a, b and c are positive integers, and ac divides bc, then a divides b.

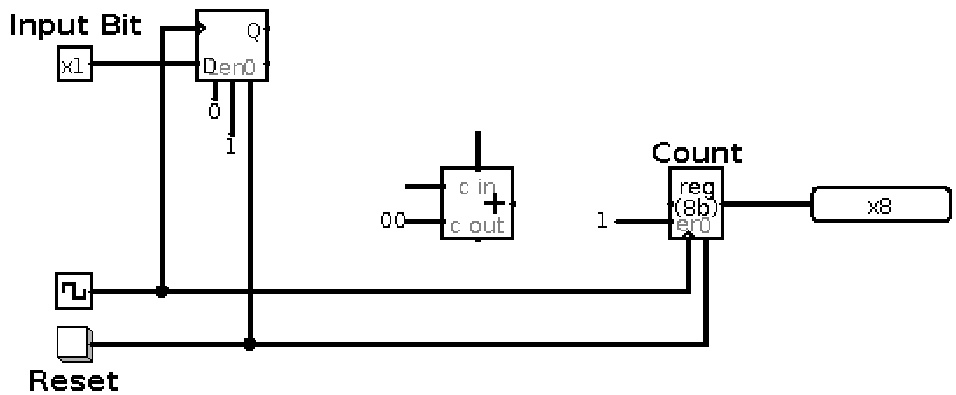
Solution : Translated into predicate logic, this is $\forall a \in \mathbf{Z}^+, \forall b \in \mathbf{Z}^+, \forall c \in \mathbf{Z}^+, ac \mid bc \rightarrow a \mid b$ (recall that $a \mid b$ means "a divides b").

To prove this, consider three unspecified positive integers a, b, c. Assume that ac divides bc. Hence $bc = (ac)x$ for some integer x. Since c is positive, we can divide both sides of the equality by c, and hence $b = ax$. Thus a divides b.

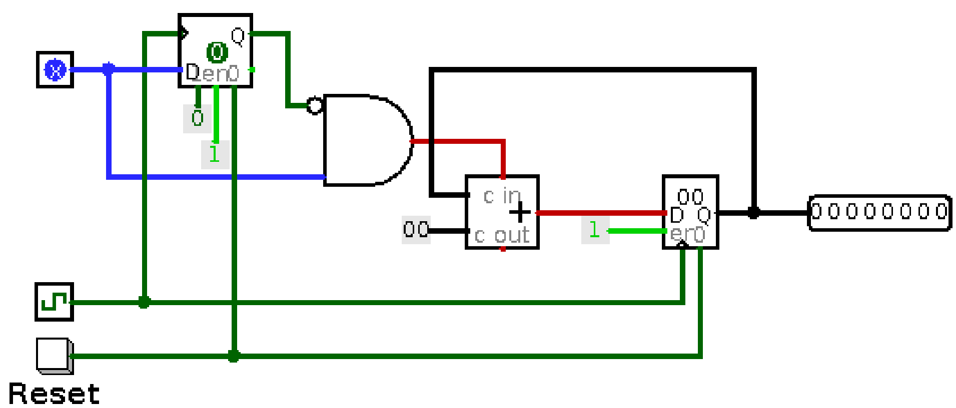
[6] 5. One of your teaching assistants designed a circuit to count the number of distinct sequences of consecutive 1 bits in a multi-bit input (the next bit of the input is available every time the clock ticks). For instance, the output of the circuit should be

- 1 if the input was 11100,
- 2 if the input was 0111101,
- 3 if the input was 10010010,

etc. Unfortunately a crazy Computer Science instructor armed with an eraser came by! Help the teaching assistant by restoring the gate(s) and/or wire(s) the instructor has deleted, to get a working circuit.



Solution :



[6] 6. Using a proof by contradiction, prove that for all positive integers a and b, $(a+b)/2 \geq \sqrt{ab}$. Hints: (1) you should square both sides of an inequality, (2) what is $(a-b)^2$?

Solution : We use a proof by contradiction (as stated in the question!). Consider two unspecified positive integers a, b, and assume that $(a+b)/2 < \sqrt{ab}$. Because $(a+b)/2$ and $\sqrt{ab}$ are both positive, this means that

$$\left(\frac{a+b}{2}\right)^2 < (\sqrt{ab})^2$$

and hence

$$\frac{a^2+2ab+b^2}{4} < ab$$

or stated otherwise

$$a^2 + 2ab + b^2 < 4ab.$$

However, by substracting $4ab$ from both sides of the inequality, this means that

$$a^2 - 2ab + b^2 < 0$$

which is the same as stating that

$$(a - b)^2 < 0$$

which is impossible (squares of integers can not be negative). Consequently $(a + b)/2 \geq \sqrt{ab}$.