CPSC 121 Sample Final Examination
April 2013

Name: ______________________ Student ID: ______________________

Signature: ______________________

Question	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

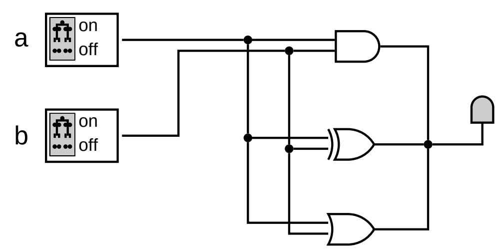
- You have 150 minutes to write the 11 questions on this examination. A total of 98 marks are available.
- **Justify all of your answers.**
- You are only allowed to bring in one hand-written (or printed using at least a 11 point font), double-sided 8.5 x 11in sheet of notes.
- Keep your answers short. If you run out of space for a question, you have written too much.
- The number in square brackets to the left of the question number indicates the number of marks allocated for that question. Use these to help you determine how much time you should spend on each question.
- Use the back of the pages for your rough work.
- **Good luck!**

UNIVERSITY REGULATIONS:

- Each candidate should be prepared to produce, upon request, his/her UBC card.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
- CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
  2. Speaking or communicating with other candidates.
  3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

[6] 1. Short answers

[3] a. What is wrong with the following circuit?



[3] b. Explain briefly the difference between the purpose of a half-adder and that of a full-adder.

[6] 2. Let $A = \{1, 2, 3 \ldots, 299, 300\}$.

[3] a. Give an example of a function f from $\mathbf{Z}^+$ into A that is not onto (surjective).

[3] b. Consider the function $f : \{1, 2, \ldots, 18\} \to A$ defined by $f(x) = x^2 - 3x + 3$. Is f one-to-one (injective)? Explain why or why not.

[9] 3. Number representation

[3] (a) What unsigned integer is represented by the 9-bit value 100111011?

[3] (b) What signed integer is represented by the 9-bit value 100111011?

[3] (c) A professional programmer wrote a program that takes a floating point number x, sets its value to 0, and then adds 0.1 to x repeatedly until it becomes equal to 10, printing each value of x as it goes. Why does the program produce the output

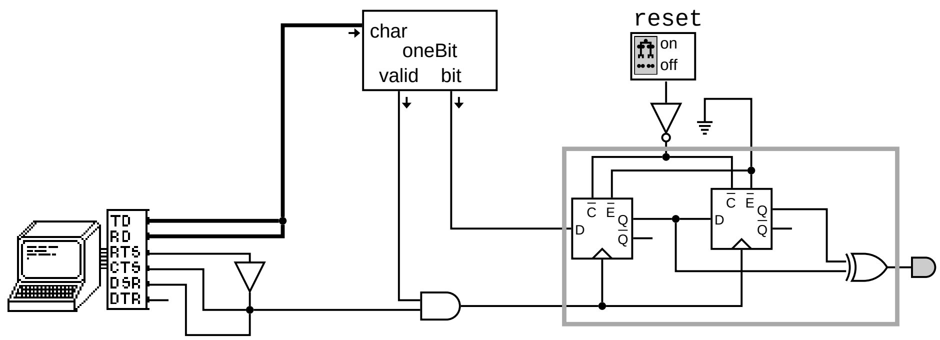
```
0.0  0.1  0.2  0.3  0.4  0.5  0.6  0.70000005  0.8000001  0.9000001
1.0000001  1.1000001  1.2000002
```

(and so on) instead of

```
0.0  0.1  0.2  0.3  0.4  0.5  0.6  0.7  0.8  0.9  1.0
```

?

[12] 4. Consider the following circuit:



The terminal and the `oneBit` subcircuit provide bits one at a time to the remainder of the circuit, and the clock attached to the D flip-flops ticks each time a bit is available.

[4] a. Suppose that the user first flips the `reset` switch on, then off, and finally types in the digits 0, 0, 1, 1, 0, 1, 1, 0 in this order. What will be the state of the LED (on/off) after each of the digits is typed (you need to provide 8 answers).

[4] b. Explain *in English* the purpose of this circuit.

[4] c. Show how to modify the part of the circuit that is inside the gray box so the LED would be **on** if an odd number of 1 digits had been typed, and **off** if an even number of 1 digits had been typed. Hint: you will only need one flip-flop, and a few other gates.

[7] 5. Determine the validity of the following argument. Justify your answer **fully**.

1. $(\sim p \vee q) \to r$
2. $r \to (s \vee t)$
3. $\sim s \wedge \sim u$
4. $\sim u \to \sim t$

$\therefore p$

[9] 6. A *perfect number* is a positive integer that is equal to the sum of its divisors (except itself). For instance, 28 is perfect since $28 = 1 + 2 + 4 + 7 + 14$. In this question, we will look at *almost-perfect numbers*: numbers that are one more than the sum of their divisors (except themselves). For instance, 16 is almost perfect since $1+2+4+8 = 15$. Consider the following statement:

> Given any positive integer n, we can always find an almost-perfect number that is larger than n.

[3] a. Translate this statement into first-order logic (that is, using predicates and quantifiers). You may assume the existence of a predicate `isAlmostPerfect`.

[6] b. Prove the statement. Hint: look carefully at the example I gave, and see if you can make it more general.

[11] 7. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and f be a function from $\mathcal{U}$ into $\mathcal{U}$. Finally, let $F : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ be the function defined by

$$F(X) = \{f(x) \mid x \in X\}$$

For instance, if $f(x) = 13 - x$, and $X = \{1, 3, 4, 7\}$, then $F(X) = \{f(1), f(3), f(4), f(7)\} = \{12, 10, 9, 6\}$.

[7] a. Prove that for every two subsets A, B of $\mathcal{U}$, we have $F(A \cap B) \subseteq F(A) \cap F(B)$.

[4] b. Give an example that shows that $F(A-B)$ is not necessarily a subset of $F(A)-F(B)$.

[8] 8. Prove that for every positive integer n,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

[8] 9. The inhabitants of the country of Pirlouitland use only three kinds of coins, worth 1 schlip, 5 schlips and 10 schlips respectively. Johan, one of the knights, makes the following observation:

> For every amount $n \geq 10$ schlips, the way to pay for n schlips that uses the fewest coins always contains a 10 schlip coin.

Prove Johan's observation using *an indirect proof*. Hint: think of substituting some coin(s) for other coin(s).

[12] 10. Mathematical Induction

[3] a. Explain briefly the two forms of mathematical induction discussed in class.

[9] b. Recall that $\lfloor x \rfloor$ is the largest integer that is $\leq x$, that $\lceil x \rceil$ is the smallest integer that is $\geq x$, and that $\forall x \in \mathbf{R}^+$, $\lfloor x/2 \rfloor + \lceil x/2 \rceil = x$.

Let $p_0, p_1, p_2, p_3, \ldots$ be a sequence of integers defined by

$$p_1 = 0,\ p_2 = 1 \qquad \text{and}$$
$$p_n = p_{\lfloor n/2 \rfloor} + p_{\lceil n/2 \rceil} + 1 \qquad \text{for all } n \in \mathbf{Z}^+ \text{ where } n \geq 3.$$

For example, $p_3 = p_{\lfloor 3/2 \rfloor} + p_{\lceil 3/2 \rceil} + 1 = p_1 + p_2 + 1 = 0 + 1 + 1 = 2$

Prove that for all nonnegative integers $n \geq 1$, $p_n = n - 1$. You must state the induction hypothesis clearly.

[10] 11. Consider the set L of all strings over the alphabet $\{a, b, c, d\}$ that start and end with the same letter, and do not contain this letter in any other position. For instance, the strings `a`, `bb` and `cabbac` belong to L, but the empty string and the strings `acccb` and `cabcbac` do not.

[7] a. Design a DFA that accepts every string in L, and reject every other string.

[3] b. Write a regular expression that matches exactly the strings that belong to L.