CPSC 121 Midterm 2
Tuesday, March 15th, 2011

[6] 1. For the first statement in each of the following pairs of statements, suggest a plausible proof approach (that is, a plan involving one or more of the proof techniques you have learned, such as proof of existence [witness proof], generalizing from the generic particular [WLOG], or direct proof [antecedent assumption]) that makes as much progress as possible on the proof. Note that you **do not** have enough information to actually prove the theorem!

For the second, slightly altered statement, explain why the statement will probably be easy to prove or disprove.

[3] a. Suggest an approach for: $\forall x \in D, P(x) \rightarrow Q(x)$

Solution : One plausible approach is to use a direct proof: we consider an unspecified element x of the domain, we assume that $P(x)$ holds, and then we show that $Q(x)$ holds using only $P(x)$ and properties that are true for every element of D (that is, we can not assume anything specific about the value of x or its properties).

Explain why this will be easy to prove/disprove: $\exists x \in D, P(x) \rightarrow Q(x)$

Solution : This theorem is true if there is at least one element x of D for which $P(x)$ is false. Hence if we can find such an element, than the theorem holds automatically (if $P(x)$ is true for every element of D, then we need to prove the theorem the hard way).

[3] b. Suggest an approach for: $\exists x \in D, \exists y \in D, (x \neq y \wedge R(x,y))$

Solution : We could use an existence proof: we choose two specific, distinct values of D (for instance, if D were the set of all integers, then we could choose -4 and 11), and then prove that $R(x,y)$ holds for the two values we have chosen.

Explain why this will be easy to prove/disprove: $\forall x \in D, \forall y \in D, (x \neq y \wedge R(x,y))$

Solution : If the domain D is empty, then this statement is vacuously true. Otherwise it is false: we can disprove it by choosing $y = x$.

[7] 2. Prove that for every distinct pair of real numbers, there is another real number that is between them (greater than the smaller one and less than the larger one). Hint: a direct proof works well.

Solution : Consider an arbitrary pair of distinct real numbers x, y, and assume without loss of generality that $x < y$ (if y was smaller than x, then we could always swap their names so x is the smaller of the two). Define $z = (x+y)/2$. Now, observe that

$$z = \frac{x+y}{2} > \frac{x+x}{2} = x$$

and that

$$z = \frac{x+y}{2} < \frac{y+y}{2} = y$$

Hence z is between x and y. QED

[6] 3. Each senior instructor in the department of Computer Science either always lies, or always tells the truth. Both Steve and Patrice are senior instructors in the department.

- Steve says: "Exactly one of us is lying."
- Patrice says: "Steve is telling the truth."

Using a proof by contradiction, show that Steve is lying.

Solution : We have two premises:

(a) Steve says: "Exactly one of us is lying."
(b) Patrice says: "Steve is telling the truth."

and we want to prove the conclusion "Steve is lying".

So assume that the two premises hold, but that Steve is telling the truth. From premise 1, this means that Patrice is lying. Therefore, from premise 2, Steve is not telling the truth. But this contradicts the negated conclusion ("Steve is telling the truth"). We thus get the contradiction we wanted, and therefore Steve is lying (note: you can then conclude that Patrice is also lying).

[8] 4. Consider the following proposition:

$$\sim\forall x \in S, (\exists y \in C, F(x, y)) \rightarrow P(x)$$

[3] a. Use the following definitions to translate the given proposition into English:

- S is the set of all students in the Faculty of Science at UBC
- C is the set of all UBC courses
- $F(x, y)$ means that student x failed course y
- $P(x)$ means that x will be put on academic probation

(For full marks, your translation must be reasonably natural-sounding, rather than a word for word translation.)

Solution : It is not true that every Science student who fails a course will be put on academic probation.

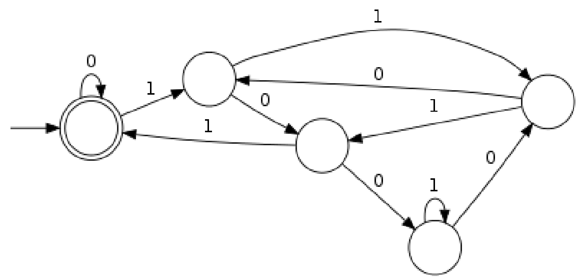
[5] b. Prove that the given proposition is logically equivalent to

$$\exists x \in S, (\exists y \in C, F(x, y)) \wedge \sim P(x)$$

Solution : Let us define $p : \sim\forall x \in S, (\exists y \in C, F(x, y)) \rightarrow P(x)$. Then

$$\begin{aligned}
p &\equiv \exists x \in S, \sim((\exists y \in C, F(x, y)) \rightarrow P(x)) && \text{by the generalized DeMorgan's law} \\
&\equiv \exists x \in S, \sim(\sim(\exists y \in C, F(x, y)) \vee P(x)) && \text{by the definition of} \rightarrow \\
&\equiv \exists x \in S, \sim\sim(\exists y \in C, F(x, y)) \wedge \sim P(x)) && \text{by the deMorgan's law} \\
&\equiv \exists x \in S, (\exists y \in C, F(x, y)) \wedge \sim P(x)) && \text{by the double negation law}
\end{aligned}$$

[7] 5. Consider the following DFA:



[5] a. Which of the following strings of bits will be accepted by the DFA (write Yes or No next to each string)?

[1] i. 1001

Solution : No.

[1] ii. 1010

Solution : Yes.

[1] iii. 1111

Solution : Yes.

[1] iv. 10101

Solution : No.

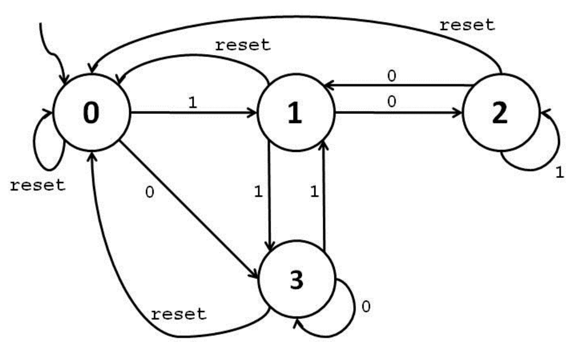
[1] v. 11001

Solution : Yes.

[2] b. Using the examples from part (a) as a guide, explain what sequences of bits will be accepted by the DFA. Hint: think of them as unsigned integers.

Solution : It accepts the strings that are unsigned integers divisible by 5, written in binary.

[8] 6. Consider the following DFA:



If the reset column in the table below is 1 then the input to the DFA is `reset` (the bit column is ignored). Otherwise, the input is the value in the bit column. We would like to implement a circuit corresponding to this DFA.

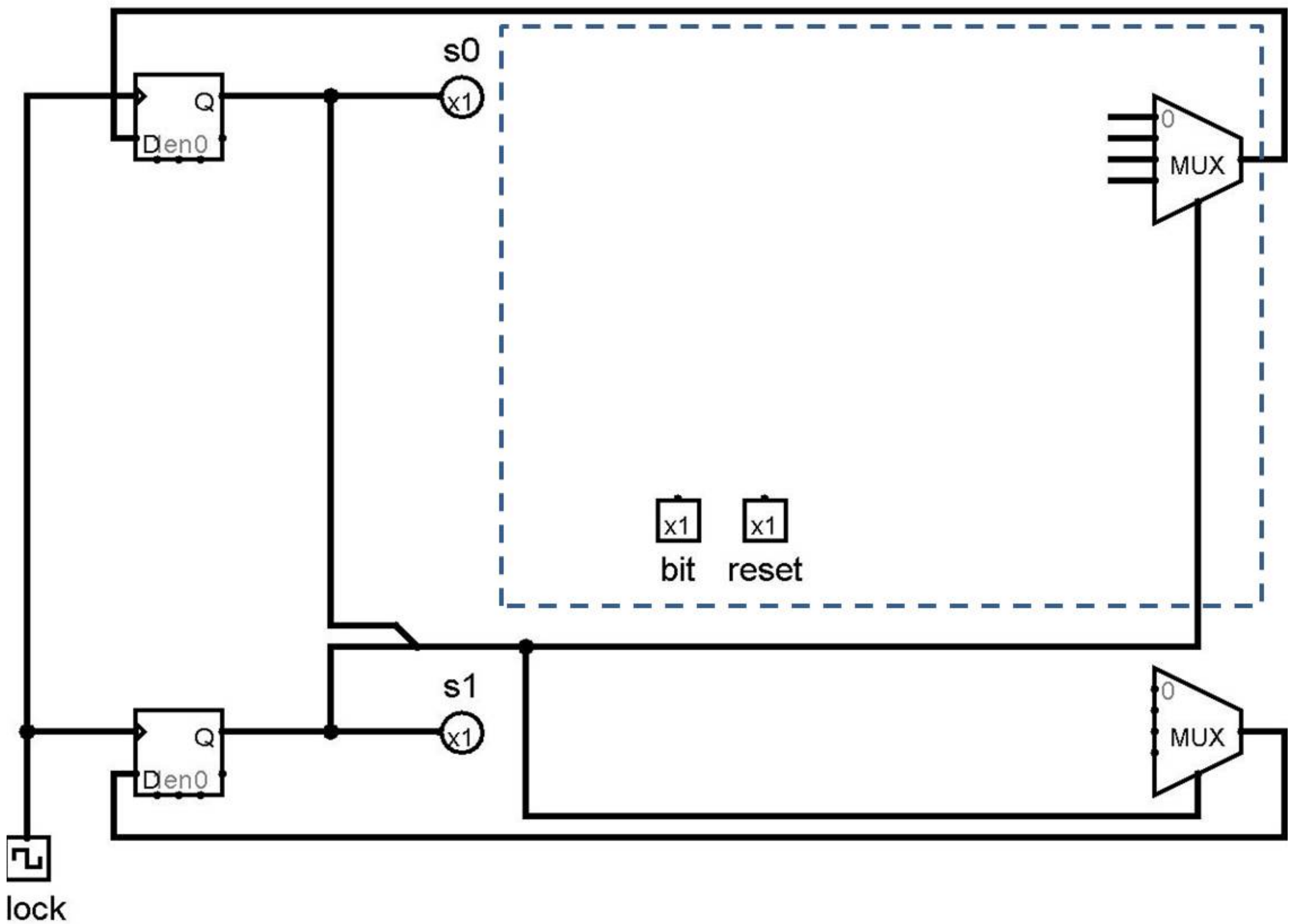
[2] a. Fill in the incomplete rows of the truth table below describing this DFA's behaviour. (The state number is included to clarify the meaning of "s0" and "s1".)

State	s0	s1	bit	reset	new s0	new s1
0	0	0	0	0		
0	0	0	0	1		
0	0	0	1	0		
0	0	0	1	1		
1	0	1	0	0		
1	0	1	0	1		
1	0	1	1	0		
1	0	1	1	1		
2	1	0	0	0	0	1
2	1	0	0	1	0	0
2	1	0	1	0	1	0
2	1	0	1	1	0	0
3	1	1	0	0	1	1
3	1	1	0	1	0	0
3	1	1	1	0	0	1
3	1	1	1	1	0	0

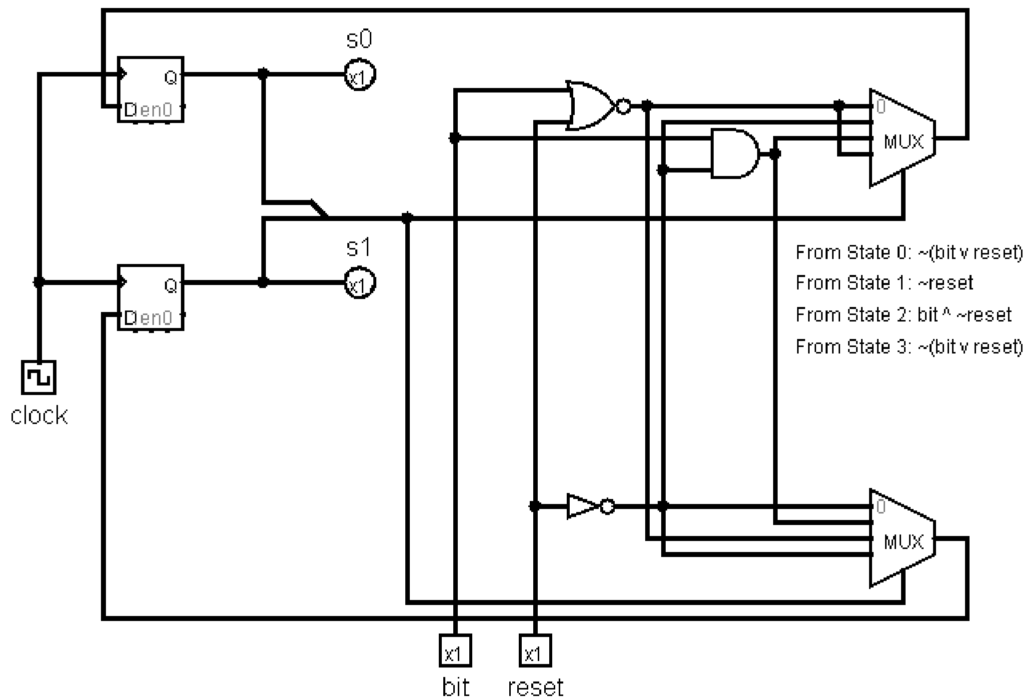
Solution:

State	s0	s1	bit	reset	new s0	new s1
0	0	0	0	0	**1**	**1**
0	0	0	0	1	**0**	**0**
0	0	0	1	0	**0**	**1**
0	0	0	1	1	**0**	**0**
1	0	1	0	0	**1**	**0**
1	0	1	0	1	**0**	**0**
1	0	1	1	0	**1**	**1**
1	0	1	1	1	**0**	**0**

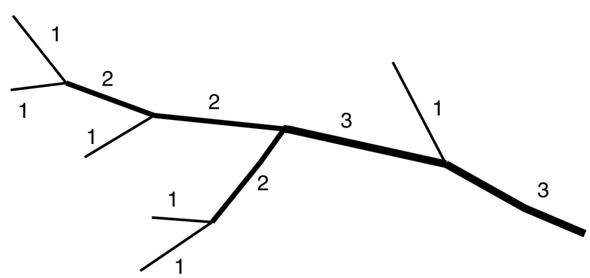
[6] b. Implement the portion of the sequential circuit below necessary to calculate the next value for the flip-flop storing s0 (i.e., the portion enclosed in the dashed rectangle).


s0
x1
Q
D
en0
0
MUX
bit
x1
reset
x1
s1
x1
Q
D
en0
0
MUX
lock

Solution :



[8] 7. This question considers a classification system for rivers. Here's a river marked up with this system:



We call the numbers SNs. To use SNs, we divide a river into segments. Each segment flows either from a starting point of the river or from where two or more other segments join. Each segment ends either where it flows into the ocean or where it joins with one or more other segments.

The SN of a segment (which is a positive integer) comes from these rules:

(a) If the segment flows from a starting point, then its SN is 1.

(b) If it flows from where two or more river segments join, then:

   i. if one of the joining segments has a SN of i and the others have smaller SNs, then the resulting segment's SN is i.

   ii. if two or more of the joining segments' SNs are i and no others have larger SNs, then the resulting segment's SN is $i + 1$.

Note: you may assume that the river system of a segment that flows from where other segments join is larger than the river systems of the segments that join together.

Problem: Prove by induction that any river segment with SN n flows from at least 2^{n-1} different starting points. (Even an incorrect proof with appropriate form will receive substantial partial credit.)

Solution : Actually, hydrologists (scientists who study water systems) use these numbers—called "Strahler numbers"—to describe river systems.

Insight (NOT part of the proof): The definition of SNs is self-referential. So, our inductive proof will likely follow that self-referential structure. Let's consider the three cases: the SN of 1 case sounds like a base case. The two cases where a segment has other segments flowing into it should work well for induction. We'll just build up a minimum number of starting points for such a segment from the minimum number of starting points for the segments that feed in (about which we know something because of the inductive hypothesis, since we know they're "smaller river systems").

Base case: A river segment that flows from a starting point of the river has a Strahler number of 1 and starts in $1 = 2^0 = 2^{1-1}$ places. So, the theorem holds.

Now, consider an arbitrary river segment R_1 that flows from the junction of two or more other segments.

Inductive Hypothesis: Assume for the (smaller) river systems that flow into the junction, that the theorem holds (i.e., for each segment that flows into the junction, if the segment's SN is i, the segment flows from at least 2^{i-1} starting points).

Inductive Step: Consider the river segment R_2 that has the highest SN k of those that flow into the junction. One of the following cases must hold:

(a) No other river segment flowing into the junction has a SN as high as k. Then, by definition, R_1 has a SN of k. By the IH, R_2 has at least 2^{k-1} starting points. Since R_2 flows into R_1, R_1 also has all those starting points—at least 2^{k-1}—and the theorem holds.

(b) Some other river segment R_3 that flows into the junction also has a SN of k. Then, by definition, R_1 has a SN of $k + 1$. By the IH, R_2 and R_3 each has at least 2^{k-1} starting points. Both flow into R_1; so, R_1 has all of these starting points—at least $2 \times 2^{k-1} = 2^k = 2^{(k+1)-1}$—and the theorem holds.

QED