CPSC 121: Models of Computation
Midterm Exam #2, 2009 November 5

Name: ______________________ Student ID: ______________________

Signature: ______________________

- You have **75 minutes** to write the 7 questions on this exam.
- A total of **50 marks** are available. You may want to complete what you consider to be the easiest questions first!
- Ensure that you clearly indicate a legible answer for each question.
- You are allowed any reasonable number of textbooks and quantity of notes as references. Otherwise, no notes, aides, or electronic equipment are allowed.
- Good luck!

UNIVERSITY REGULATIONS

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
    - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
    - speaking or communicating with other candidates; and
    - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
10	5	5	4	7	12	7	

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If you put solutions here or anywhere other than the blank provided for each solution, you must *clearly* indicate which problem the solution goes with and also indicate where the solution is at the designated area for that problem's solution.

1 Numbers and Circuits [10 marks]

Computers represent everything, including letters, as binary values. In the most commonly used representation, the letters '0', '1', '2', '3', '4', '5', '6', '7', '8', and '9' are represented by the numbers 48, 49, 50, 51, 52, 53, 54, 55, 56, and 57, respectively.

1. Give the 8-bit binary value used to represent '0'. **[2 marks]**

Powers of two reminder: In case they're handy for the question above, here are several powers of two:

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8
1	2	4	8	16	32	64	128	256

2. Which of the following are good reasons to use these values for the letters '0' through '9'? **Circle all correct answers. [2 marks]**

   (a) So that the letters '0' through '9' will have the same binary representation as the numbers 0 through 9.

   (b) So that the difference between two letters (e.g., '7' and '3') will be the same as the difference between the corresponding numbers (e.g., $7 - 3$).

   (c) No other choice would be possible since these "letters" are in fact numbers.

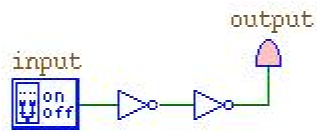
   (d) No other choice would be possible since $48 = 32 + 16$.

   (e) So that the binary representation of each letter will end with the binary representation of the corresponding number.

   (f) Because other bit patterns must be reserved for negative values.

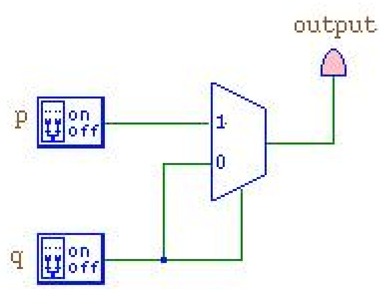
3. The uppercase letters 'A'–'Z' and the lowercase letters 'a'–'z' are also represented by consecutive numbers, but rather than being right next to each other, six symbols unrelated to the alphabet are included after the 26 uppercase letters but before the lowercase letters. Hypothesize why those six other symbols are included between the uppercase and lowercase letters. **[2 marks]**

4. Moving on to circuts problems. . . Consider the following circuit: **[2 marks]**



According to our propositional logic model of circuits, this circuit doesn't do anything or, more precisely, does the same thing as a plain wire. However, the circuit is actually used productively in real systems. What does this circuit do that propositional logic fails to model?

5. Draw a truth table for the following circuit, and indicate what two-input gate it's equivalent to, if any. **[2 marks]**



2 Abstract Proof [5 marks]

Consider the theorem: $\forall y \in D, \exists z \in D, (P(y) \vee Q(y, z)) \rightarrow (R(y) \vee S(y, z))$.

You can neither prove nor disprove this theorem without more information about the domains and predicates it uses. In this problem, you'll propose plausible proof strategies you might use if you had that additional information. Be specific about how much or little you can assume!

1. Describe a strategy to prove this theorem true without using indirect proof techniques (e.g., proof by contrapositive or contradiction). **[3 marks]**

2. How would you prove this theorem false using proof by contrapositive? **[2 marks]**

3 Glossy Predicates [5 marks]

Recall that $Elt(a, i, v)$ means that the list a has the integer value v at index i, where indexes are positive integers. Assume a list has exactly one length, exactly one value at each index up to its length, and no values at indexes greater than its length.

Two of the following definitions of the predicate $Sorted(list)$ mean "the list, which may contain duplicate values, is in sorted order from smallest to largest". Each of the others *almost* means that but not quite. Match up each statement with its meaning by writing the meaning's letter in the statement's blank. (The correct solution uses "(d)" twice and everything else once.) **[5 marks, 1 per correct matching]**

Statements:

____ 1. $\forall i \in \mathbf{Z}^+, \forall v_1 \in \mathbf{Z}, \forall v_2 \in \mathbf{Z}, \forall v_3 \in \mathbf{Z}, (Elt(list, i-1, v_1) \wedge Elt(list, i, v_2) \wedge Elt(list, i+1, v_3)) \rightarrow (v_1 \leq v_2 \leftrightarrow v_2 \leq v_3)$.

____ 2. $\sim\exists i_1 \in \mathbf{Z}^+, \exists v_1 \in \mathbf{Z}, \exists i_2 \in \mathbf{Z}^+, \exists v_2 \in \mathbf{Z}, Elt(list, i_1, v_1) \wedge Elt(list, i_2, v_2) \wedge i_1 < i_2 \wedge v_1 > v_2$.

____ 3. $\forall i_1 \in \mathbf{Z}^+, \forall v_1 \in \mathbf{Z}, \forall i_2 \in \mathbf{Z}^+, \forall v_2 \in \mathbf{Z}, (Elt(list, i_1, v_1) \wedge Elt(list, i_2, v_2) \wedge i_1 < i_2) \rightarrow v_1 < v_2$.

____ 4. $\forall i_1 \in \mathbf{Z}^+, \forall v_1 \in \mathbf{Z}, \forall i_2 \in \mathbf{Z}^+, \forall v_2 \in \mathbf{Z}, (Elt(list, i_1, v_1) \wedge Elt(list, i_2, v_2) \wedge v_1 \neq v_2) \rightarrow (i_1 > i_2 \leftrightarrow v_1 > v_2)$.

____ 5. $\exists i_1 \in \mathbf{Z}^+, \exists v_1 \in \mathbf{Z}, \forall i_2 \in \mathbf{Z}^+, \forall v_2 \in \mathbf{Z}, Elt(list, i_1, v_1) \wedge (Elt(list, i_2, v_2) \rightarrow (i_1 > i_2 \leftrightarrow v_1 > v_2))$.

Meanings:

(a) The list is sorted from smallest to largest but contains no duplicate values.

(b) The list is sorted either from smallest to largest (increasing) or largest to smallest (decreasing), where only the increasing order allows duplicates.

(c) The list is "partitioned"—everything to the left of one of the list elements is less than or equal to it and everything to the right is greater than it—but not necessarily sorted.

(d) The list, which may contain duplicate values, is in sorted order from smallest to largest. *This meaning is used twice in the correct solution.*

4 Propositional Logic Proof [4 marks]

Complete the following formal propositional logic proof that $\sim y$ follows from the premises listed below. Use only logical equivalence laws and rules of inference in Epp Chapter 1 plus our "definition of conditional" and "definition of biconditional". You need not explicitly show steps that rely only on the double negation, commutative, and associative logical equivalence rules.

1.	$\sim x \vee z \vee r$	premise
2.	$y \rightarrow (\sim q \wedge \sim r)$	premise
3.	$u \wedge x$	premise
4.	$p \rightarrow (q \vee r)$	premise
5.	$\sim(\sim p \wedge u)$	premise

5 A Little Room for Proof [7 marks]

Consider the theorem: $\forall x \in D, \lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$.

D contains only positive integers although it does not necessarily contain all the positive integers. (For example, it might be only powers of 5: 5, 25, 125, 625, ...)

1. Without more information about the set D, which of the following are plausible approaches that could prove this statement ***true***? **Circle all correct answers. [2 marks]**

   (a) Select a particular element of D, and show that $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$ when x is that element.

   (b) Without loss of generality, let x be an element of D, assume $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil$, and then prove x.

   (c) Assume that $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil \neq x$ for some $x \in D$, and show that that leads to a contradiction.

   (d) Without loss of generality, let x be an element of D, assume $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$, and show that that leads to a contradiction.

   (e) Without loss of generality, let x be an element of D, break the entire domain D into parts, and then prove $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$ holds when x comes from each part. (Depending on D, the "parts" might be something like numbers of the form $3k$, $3k + 1$, and $3k + 2$, where k is an integer.)

2. When $D = \{3, 4, 5\}$, this statement is not true. Why not? **[1 mark]**

   (a) Because it is untrue when $x = 3$.

   (b) Because it is untrue when $x = 4$.

   (c) Because it is untrue when $x = 5$.

   (d) None of these, although a single example can disprove the statement.

   (e) None of these, because no single example can disprove the statement.

3. The statement is true when D is the set of all positive integers that are one more than a multiple of three—numbers like 1, 4, 7, 10, ..., of the form $x = 3k + 1$, for a non-negative integer k. Prove the statement true for this domain. **[4 marks]**

6 Peer-educate Logic [12 marks]

CPSC has decided that students should have "peer mentors". We use the predicate $Mentor(x, y)$ to indicate that x's mentor is y. To make it easy to start the program, we have each student in CPSC 121 select another student in CPSC 121 to be their mentor. (Let C be the set of all CPSC 121 students.)

1. Translate each of the following into predicate logic.

   (a) Every 121 student has a 121 student as their mentor. **[2 marks]**

   (b) Every 121 student has a 121 student **other than themselves** as a mentor. **[2 marks]**

   (c) Every 121 student has **exactly one** 121 student as a mentor. **[2 marks]**

2. Explain in English what the following predicate means. (Note: For this part, students may have more than one mentor.) **[2 marks]**

   $Pair(x, y) \equiv x \neq y \ \wedge \ \forall z \in C, (Mentor(z, x) \vee Mentor(z, y)) \rightarrow (Mentor(z, x) \wedge Mentor(z, y))$.

3. If we can start with one person and then trace through their mentors to get back to the same person, we call that a "mentor cycle". (For example, if one person's mentor's mentor's mentor's mentor is themselves, that forms a mentor cycle.)

   Prove by contradiction that if there are $n \geq 2$ students in CPSC 121 and every student has exactly one other CPSC 121 student as their mentor, then there is a mentor cycle. **[4 marks]**

   Hint: imagine my mentor and I are two different people. If my mentor's mentor isn't me or my mentor, then together we must be ***three*** *different people. If we carry this argument on, how many people are in the class?*

7 Hare-Racing Experience [7 marks]

Tortoise and Hare are racing. Tortoise moves one metre every minute. Hare moves $2n - 1$ metres on the n^{th} minute (which is equivalent to n^2 metres in the first n minutes).

1. Disprove the following statement: If Tortoise starts 10 metres ahead, Hare will have caught up after 2 minutes. **[1 mark]**

2. Prove the following statement: If Tortoise starts 10 metres ahead, Hare will have caught up after $2 * 10 = 20$ minutes. **[1 mark]**

3. Prove that no matter how many metres ahead Tortoise starts, Hare will catch up given enough time. **[5 marks]**

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