CPSC 121: Models of Computation
Midterm Exam #2, 2009 November 5

Solution : **SAMPLE SOLUTION**

Name: ______________________ Student ID: ______________________

Signature: ______________________

- You have **75 minutes** to write the 7 questions on this exam.
- A total of **50 marks** are available. You may want to complete what you consider to be the easiest questions first!
- Ensure that you clearly indicate a legible answer for each question.
- You are allowed any reasonable number of textbooks and quantity of notes as references. Otherwise, no notes, aides, or electronic equipment are allowed.
- Good luck!

UNIVERSITY REGULATIONS

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   - speaking or communicating with other candidates; and
   - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
10	5	5	4	7	12	7	

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If you put solutions here or anywhere other than the blank provided for each solution, you must *clearly* indicate which problem the solution goes with and also indicate where the solution is at the designated area for that problem's solution.

1 Numbers and Circuits [10 marks]

Computers represent everything, including letters, as binary values. In the most commonly used representation, the letters '0', '1', '2', '3', '4', '5', '6', '7', '8', and '9' are represented by the numbers 48, 49, 50, 51, 52, 53, 54, 55, 56, and 57, respectively.

1. Give the 8-bit binary value used to represent '0'. **[2 marks]**

Solution : $48 = 32 + 16 = 2^5 + 2^4 = 00110000_2$

Powers of two reminder: In case they're handy for the question above, here are several powers of two:

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8
1	2	4	8	16	32	64	128	256

2. Which of the following are good reasons to use these values for the letters '0' through '9'? **Circle all correct answers. [2 marks]**

(a) So that the letters '0' through '9' will have the same binary representation as the numbers 0 through 9.

Solution : Definitely not. For example, as we've seen, '0' has the binary representation 00110000, not 00000000.

(b) So that the difference between two letters (e.g., '7' and '3') will be the same as the difference between the corresponding numbers (e.g., $7 - 3$).

Solution : This is correct—consider that '1' has a numeric value one greater than '0' and the same is true for each subsequent letter—and (potentially) valuable.

(c) No other choice would be possible since these "letters" are in fact numbers.

Solution : There's no law that this is how the numbers must be laid out. For a concrete example, EBCDIC uses different values.

(d) No other choice would be possible since $48 = 32 + 16$.

Solution : There's no law that this is how the numbers must be laid out. For a concrete example, EBCDIC uses different values.

(e) So that the binary representation of each letter will end with the binary representation of the corresponding number.

Solution : This is true and potentially handy. (Why is it true? Because the representation for '0' ends with 0000, and the representation for the numbers 0–9 all fit in those four bits.)

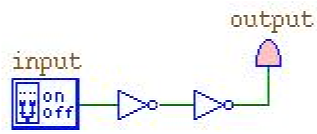
(f) Because other bit patterns must be reserved for negative values.

Solution : This is not a meaningful answer (and so is wrong).

3. The uppercase letters 'A'–'Z' and the lowercase letters 'a'–'z' are also represented by consecutive numbers, but rather than being right next to each other, six symbols unrelated to the alphabet are included after the 26 uppercase letters but before the lowercase letters. Hypothesize why those six other symbols are included between the uppercase and lowercase letters. **[2 marks]**

   Solution : There may be other valid responses, but consider that $26+6=32$, and since $32=2^5$, the uppercase and lowercase representations will differ in only one bit.

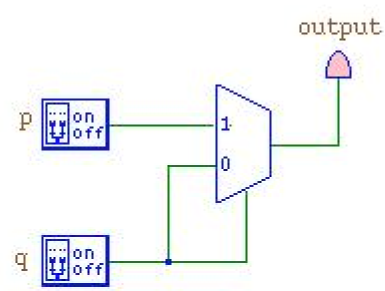
4. Moving on to circuts problems... Consider the following circuit: **[2 marks]**



According to our propositional logic model of circuits, this circuit doesn't do anything or, more precisely, does the same thing as a plain wire. However, the circuit is actually used productively in real systems. What does this circuit do that propositional logic fails to model?

Solution : Propositional logic does not model the time it takes for gates to calculate ("propagate") their results. This is a delay circuit that takes a signal and "slows it down" as it proceeds through the circuit. This can be very useful in, e.g., the sequential circuits we've just begun studying where timing can be quite important.

5. Draw a truth table for the following circuit, and indicate what two-input gate it's equivalent to, if any. **[2 marks]**



Solution :

p	q	$output$
T	T	T
T	F	F
F	T	F
F	F	F

This is the truth table for OR. So, this circuit is equivalent to an OR gate.

(Note that our symbol for a multiplexer reminds you which input line is active when the control line is 0 vs. 1. We've used some versions where the top line is the "1" line and others where the top line is the "0" line, but as long as the lines are explicitly labeled, the gate is unambiguous.)

2 Abstract Proof [5 marks]

Consider the theorem: $\forall y \in D, \exists z \in D, (P(y) \vee Q(y, z)) \to (R(y) \vee S(y, z))$.

You can neither prove nor disprove this theorem without more information about the domains and predicates it uses. In this problem, you'll propose plausible proof strategies you might use if you had that additional information. Be specific about how much or little you can assume!

1. Describe a strategy to prove this theorem true without using indirect proof techniques (e.g., proof by contrapositive or contradiction). **[3 marks]**

Solution : Without loss of generality, consider a y in D.

Choose a particular value for z from D on the basis of y.

Assume [for antecedent assumption] that $P(y) \vee Q(y, z)$ is true. (Many people also pointed out that you might approach this $\vee$ using a proof by cases.)

Prove $R(y) \vee S(y, z)$ (e.g., by proving one or the other of the two facts).

2. How would you prove this theorem false using proof by contrapositive? **[2 marks]**

Solution : This question had a typo that we did not discover until marking the exam. We therefore removed the question from the exam, although we did give extra credit for (roughly) identifying the contrapositive of the original statement.

(As is, the question is broken since the negation of the conditional and its contrapositive are identical, but for reordering of terms with the commutative law. So, it's not at all clear what using the contrapositive would mean.)

The question was intended to ask how to prove the theorem *false* using proof by contradiction, which would involve assuming the theorem is *true* and deriving a contradiction.

To account for the absence of this question, we are adding two marks to everyone's score and retaining the total of 50 for the exam (although the marked copies do not include the $+2$ and say the exam is out of 48 instead).

3 Glossy Predicates [5 marks]

Recall that $Elt(a, i, v)$ means that the list a has the integer value v at index i, where indexes are positive integers. Assume a list has exactly one length, exactly one value at each index up to its length, and no values at indexes greater than its length.

Two of the following definitions of the predicate $Sorted(list)$ mean "the list, which may contain duplicate values, is in sorted order from smallest to largest". Each of the others *almost* means that but not quite. Match up each statement with its meaning by writing the meaning's letter in the statement's blank. (The correct solution uses "(d)" twice and everything else once.) **[5 marks, 1 per correct matching]**

Statements:

____ 1. $\forall i \in \mathbf{Z}^+, \forall v_1 \in \mathbf{Z}, \forall v_2 \in \mathbf{Z}, \forall v_3 \in \mathbf{Z}, (Elt(list, i-1, v_1) \wedge Elt(list, i, v_2) \wedge Elt(list, i+1, v_3)) \rightarrow (v_1 \leq v_2 \leftrightarrow v_2 \leq v_3)$.

Solution: (b) this says, effectively, that if we consider three neighboring values, they either go up or down. (Either the middle is larger than the first and the last is larger than the middle or for both comparisons the opposite is true.)

____ 2. $\sim \exists i_1 \in \mathbf{Z}^+, \exists v_1 \in \mathbf{Z}, \exists i_2 \in \mathbf{Z}^+, \exists v_2 \in \mathbf{Z}, Elt(list, i_1, v_1) \wedge Elt(list, i_2, v_2) \wedge i_1 < i_2 \wedge v_1 > v_2$.

Solution: (d) this says, effectively, that there's no pair of elements that are *out* of order. (So, everything is in order.)

____ 3. $\forall i_1 \in \mathbf{Z}^+, \forall v_1 \in \mathbf{Z}, \forall i_2 \in \mathbf{Z}^+, \forall v_2 \in \mathbf{Z}, (Elt(list, i_1, v_1) \wedge Elt(list, i_2, v_2) \wedge i_1 < i_2) \rightarrow v_1 < v_2$.

Solution: (a) this says, effectively, that every pair of elements is in order, but insists that the left one be *strictly smaller* than the right.

____ 4. $\forall i_1 \in \mathbf{Z}^+, \forall v_1 \in \mathbf{Z}, \forall i_2 \in \mathbf{Z}^+, \forall v_2 \in \mathbf{Z}, (Elt(list, i_1, v_1) \wedge Elt(list, i_2, v_2) \wedge v_1 \neq v_2) \rightarrow (i_1 > i_2 \leftrightarrow v_1 > v_2)$.

Solution: (d) this says, effectively, that if we consider a pair of dissimilar elements, one's to the right of the other if and only if it's larger.

____ 5. $\exists i_1 \in \mathbf{Z}^+, \exists v_1 \in \mathbf{Z}, \forall i_2 \in \mathbf{Z}^+, \forall v_2 \in \mathbf{Z}, Elt(list, i_1, v_1) \wedge (Elt(list, i_2, v_2) \rightarrow (i_1 > i_2 \leftrightarrow v_1 > v_2))$.

Solution: (c) this one talks about *one* element that exists and that divides the rest of the list up. (Note the quantifiers!)

Meanings:

(a) The list is sorted from smallest to largest but contains no duplicate values.

(b) The list is sorted either from smallest to largest (increasing) or largest to smallest (decreasing), where only the increasing order allows duplicates.

(c) The list is "partitioned"—everything to the left of one of the list elements is less than or equal to it and everything to the right is greater than it—but not necessarily sorted.

(d) The list, which may contain duplicate values, is in sorted order from smallest to largest. *This meaning is used twice in the correct solution.*

4 Propositional Logic Proof [4 marks]

Complete the following formal propositional logic proof that $\sim y$ follows from the premises listed below. Use only logical equivalence laws and rules of inference in Epp Chapter 1 plus our "definition of conditional" and "definition of biconditional". You need not explicitly show steps that rely only on the double negation, commutative, and associative logical equivalence rules.

1.	$\sim x \vee z \vee r$	premise
2.	$y \rightarrow (\sim q \wedge \sim r)$	premise
3.	$u \wedge x$	premise
4.	$p \rightarrow (q \vee r)$	premise
5.	$\sim(\sim p \wedge u)$	premise

Solution :

6.	$p \vee \sim u$	De Morgan's on 5
7.	u	simplification on 3
8.	p	elimination on 6 and 7
9.	$q \vee r$	modus ponens on 8 and 4
10.	$\sim(\sim q \wedge \sim r)$	De Morgan's on 9
11.	$\sim y$	Modus Tollens on 10 and 2

5 A Little Room for Proof [7 marks]

Consider the theorem: $\forall x \in D, \lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$.

D contains only positive integers although it does not necessarily contain all the positive integers. (For example, it might be only powers of 5: 5, 25, 125, 625, ...)

1. Without more information about the set D, which of the following are plausible approaches that could prove this statement ***true***? **Circle all correct answers. [2 marks]**

   (a) Select a particular element of D, and show that $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$ when x is that element.

   Solution : No. This could prove an existential, but not a universal.

   (b) Without loss of generality, let x be an element of D, assume $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil$, and then prove x.

   Solution : No. It's no more meaningful to "assume $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil$" than it would be to "assume 12". Neither "assumption" is a proposition (something that is either true or false); so, how do we assume it's true?

   (c) Assume that $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil \neq x$ for some $x \in D$, and show that that leads to a contradiction.

   Solution : Yes, a perfectly good strategy.

   (d) Without loss of generality, let x be an element of D, assume $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$, and show that that leads to a contradiction.

   Solution : No, this would *disprove* the theorem.

   (e) Without loss of generality, let x be an element of D, break the entire domain D into parts, and then prove $\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil = x$ holds when x comes from each part. (Depending on D, the "parts" might be something like numbers of the form $3k$, $3k + 1$, and $3k + 2$, where k is an integer.)

   Solution : Yes, a perfectly good strategy.

2. When $D = \{3, 4, 5\}$, this statement is not true. Why not? **[1 mark]**

   (a) Because it is untrue when $x = 3$.

   (b) Because it is untrue when $x = 4$.

   (c) Because it is untrue when $x = 5$.

   (d) None of these, although a single example can disprove the statement.

   (e) None of these, because no single example can disprove the statement.

   Solution : Because it is untrue when $x = 5$. Consider: $\lfloor 5/3 \rfloor + \lfloor 5/3 \rfloor + \lceil 5/3 \rceil = \lfloor 1 + 2/3 \rfloor + \lfloor 1 + 2/3 \rfloor + \lceil 1 + 2/3 \rceil = 1 + 1 + 2 = 4 \neq 5$.

3. The statement is true when D is the set of all positive integers that are one more than a multiple of three—numbers like 1, 4, 7, 10, ..., of the form $x = 3k + 1$, for a non-negative integer k. Prove the statement true for this domain. **[4 marks]**

Solution : This is very similar to the homework problem about floors and ceilings.

Without loss of generality, let $x = 3k + 1$ for some non-negative integer k. Then:

$$\begin{aligned}
\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil &= \lfloor (3k+1)/3 \rfloor + \lfloor (3k+1)/3 \rfloor + \lceil (3k+1)/3 \rceil \\
&= \lfloor k + 1/3 \rfloor + \lfloor k + 1/3 \rfloor + \lceil k + 1/3 \rceil \\
&= k + k + (k+1) \\
&= 3k + 1 \\
&= x
\end{aligned}$$

QED

Note: it is generally incorrect and definitely bad form to write the proof instead like this:

$$\begin{aligned}
\lfloor x/3 \rfloor + \lfloor x/3 \rfloor + \lceil x/3 \rceil &= x \\
\lfloor (3k+1)/3 \rfloor + \lfloor (3k+1)/3 \rfloor + \lceil (3k+1)/3 \rceil &= x \\
\lfloor k + 1/3 \rfloor + \lfloor k + 1/3 \rfloor + \lceil k + 1/3 \rceil &= x \\
k + k + (k+1) &= x \\
3k + 1 &= x \\
x &= x
\end{aligned}$$

Why? This “proof” begins by asserting the statement we’re trying to prove.

It’s kind of like me saying: “I’m going to prove that CPSC 121 is a great class. Let’s start by assuming CPSC 121 is a great class. Great classes are the ones you come out of saying “wow!” We know from our assumption that you come out of CPSC 121 saying “wow!” But that means CPSC 121 is a great class. QED

We can make a lot of progress assuming something very much *like* our theorem and using appropriate proof techniques (such as proof by contradiction and, soon, induction), but just assuming the theorem is correct isn’t useful.

6 Peer-educate Logic [12 marks]

CPSC has decided that students should have "peer mentors". We use the predicate $Mentor(x, y)$ to indicate that x's mentor is y. To make it easy to start the program, we have each student in CPSC 121 select another student in CPSC 121 to be their mentor. (Let C be the set of all CPSC 121 students.)

1. Translate each of the following into predicate logic.

   (a) Every 121 student has a 121 student as their mentor. **[2 marks]**

   Solution: $\forall s_1 \in C, \exists s_2 \in C, Mentor(s_1, s_2)$

   (b) Every 121 student has a 121 student **other than themselves** as a mentor. **[2 marks]**

   Solution: $\forall s_1 \in C, \exists s_2 \in C, s_1 \neq s_2 \wedge Mentor(s_1, s_2)$
   A lovely alternate solution is:
   $\forall s_1 \in C, \exists s_2 \in C, Mentor(s_1, s_2) \wedge \sim Mentor(s_1, s_2)$

   (c) Every 121 student has **exactly one** 121 student as a mentor. **[2 marks]**

   Solution: $\forall s_1 \in C, \exists s_2 \in C, Mentor(s_1, s_2) \wedge \forall s_3 \in C, Mentor(s_1, s_3) \rightarrow s_2 = s_3$
   An alternate, logically equivalent solution is:
   $\forall s_1 \in C, \exists s_2 \in C, Mentor(s_1, s_2) \wedge \sim \exists s_3 \in C, Mentor(s_1, s_3) \wedge s_2 \neq s_3$

2. Explain in English what the following predicate means. (Note: For this part, students may have more than one mentor.) **[2 marks]**

   $Pair(x, y) \equiv x \neq y \ \wedge \ \forall z \in C, (Mentor(z, x) \vee Mentor(z, y)) \rightarrow (Mentor(z, x) \wedge Mentor(z, y)).$

   Solution: $Pair(x, y)$ means x and y are distinct CPSC 121 students, and all of their "mentees" are shared. That is, one of them mentors a student if and only if the other one also mentors that student.

   Notice that our solution explains the relationship between x and y if $Pair(x, y)$ is true. That's because the predicate logic on the right of the "$\equiv$" above is *not* a statement. It has unbound variables. So, the meaning is not an assertion about something that's true but a relationship description like this one.

3. If we can start with one person and then trace through their mentors to get back to the same person, we call that a "mentor cycle". (For example, if one person's mentor's mentor's mentor's mentor's mentor is themselves, that forms a mentor cycle.)

   Prove by contradiction that if there are $n \geq 2$ students in CPSC 121 and every student has exactly one other CPSC 121 student as their mentor, then there is a mentor cycle. **[4 marks]**

   Hint: imagine my mentor and I are two different people. If my mentor's mentor isn't me or my mentor, then together we must be ***three*** *different people. If we carry this argument on, how many people are in the class?*

Solution : Assume [for contradiction] that in a class of size n (with $n \geq 2$) in which every student has exactly one other CPSC 121 student as their mentor, there are no mentor cycles in the class.

Consider an abitrary student in the class. They have a mentor (by assumption). That mentor cannot be the student themselves. So, it's another student. Between them, that makes two distinct students in the class. The mentor must, in turn, have a mentor who cannot be either of the two students so far (or there would be a cycle), which makes three distinct students in the class. The third student must have a fourth student as a mentor who, again, cannot be any of the original three or there would be a cycle. If we repeat this process enough times, we will discover that there are $n + 1$ distinct students in the class, which is a contradiction with the fact that there are only n students in the class. QED

7 Hare-Racing Experience [7 marks]

Tortoise and Hare are racing. Tortoise moves one metre every minute. Hare moves $2n - 1$ metres on the n^{th} minute (which is equivalent to n^2 metres in the first n minutes).

1. Disprove the following statement: If Tortoise starts 10 metres ahead, Hare will have caught up after 2 minutes. **[1 mark]**

   Solution : Note that this is *not* a quantified statement. It's just an assertion about a concrete state of the world. Will Hare be ahead after 2 minutes? We can answer that question with a bit of arithmetic. This is *not* the place for your predicate logic proof strategies!

   After 2 minutes, Hare will have travelled $2^2 = 4$ metres. Tortoise will have travelled 2 metres. Assuming Tortoise starts 10 metres ahead, Hare's location will be 4, and Tortoise's location will be $10 + 2 = 12 > 4$; so, Hare will not have caught up.

2. Prove the following statement: If Tortoise starts 10 metres ahead, Hare will have caught up after $2 * 10 = 20$ minutes. **[1 mark]**

   Solution : As above, we can approach this with nothing but arithmetic.

   After 20 minutes, Hare will have travelled $20^2 = 400$ metres. Tortoise will have travelled 20 metres. Assuming Tortoise starts 10 metres ahead, Hare's location will be 400, and Tortoise's location will be $10+20 = 30 < 400$; so, Hare will have caught up.

3. Prove that no matter how many metres ahead Tortoise starts, Hare will catch up given enough time. **[5 marks]**

   Solution : This one is in predicate logic and is very similar to the assignment question about n^2 vs. $c2n$.

   There are many ways to prove this, but taking our cue from the previous part, ...

   Without loss of generality, let Tortoise start x metres ahead. Then, if the race takes $2x$ minutes, Tortoise will have gone $3x$ metres ($2x$ metres plus his x metre head start). Hare will have gone $(2x)^2 = 4x^2 \geq 4x > 3x$ metres... *if* $x > 0$. If $x = 0$, then $4x \not> 3x$. However, if $x = 0$, then after 2 minutes, Hare will have passed Tortoise (4 metres vs. 2 metres). Either way, Hare will have caught up given enough time. QED.

   (We write the proof this way to emphasize the easy-to-overlook $x = 0$ case. You could rewrite it after discovering that case either explicitly as a proof by cases or by making the time something like $2x + 2$ minutes, where $(2x+2)^2$ really is always greater than $3x+2$ for any real valued x: $(2x+2)^2 = 4x^2+8x+4 > 4x+8x+4 = 12x+4 > 3x+2$.

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