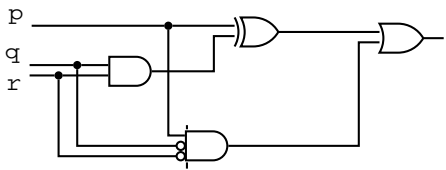
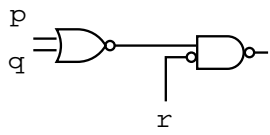


CPSC 121 Sample Final Examination  
December 2008

- [16] 1. { **Medium-Easy** } Consider the following two columns containing logical expressions, set expressions, and circuits. Each element of the first column is equivalent to an element of the second column. For each element of the first column, write down the element of the second column that it is equivalent to, and give a short proof to justify your choice.

A. $(P \Delta Q) \cup R$	1. $p \oplus (q \wedge r)$
B. $p \vee q \vee r$	2. $r \vee (\sim q \rightarrow \sim p)$
C. 	3. 
D. $(p \rightarrow q) \vee \sim p \vee r$	4. $(P \cup Q \cup R) - (P \cap Q \cap R^C)$

**Solution:**

- A goes with 4 (these were the only two expressions involving sets):

$$\begin{aligned}
 (P \Delta Q) \cup R &= ((P \cup Q) - (P \cap Q)) \cup R \\
 &= ((P \cup Q) \cap (P \cap Q)^C) \cup R \\
 &= ((P \cup Q) \cup R) \cap ((P \cap Q)^C \cup R) \\
 &= ((P \cup Q) \cup R) - ((P \cap Q)^C \cup R)^C \\
 &= ((P \cup Q) \cup R) - ((P \cap Q) \cap R^C) \\
 &= (P \cup Q \cup R) - (P \cap Q \cap R^C)
 \end{aligned}$$

- B goes with 3. The circuit for (3) outputs the value  $\sim(\sim(p \vee q) \wedge \sim r)$ , and

$$\begin{aligned}
 \sim(\sim(p \vee q) \wedge \sim r) &\equiv (\sim\sim(p \vee q) \vee \sim\sim r) \\
 &\equiv (p \vee q) \vee r
 \end{aligned}$$

- C goes with 1. The circuit for (C) outputs  $(p \oplus (q \wedge r)) \vee (p \wedge \sim q \wedge \sim r)$ . Observe however that if  $(p \wedge \sim q \wedge \sim r)$  is true, then  $p \oplus (q \wedge r)$  is also true. Therefore the term  $(p \wedge \sim q \wedge \sim r)$  does not change the truth value produced by the circuit, which is thus  $p \oplus (q \wedge r)$ .

- D goes with 2.

$$\begin{aligned}
 (p \rightarrow q) \vee \sim p \vee r &\equiv (\sim p \vee q) \vee \sim p \vee r \\
 &\equiv \sim p \vee q \vee r \\
 &\equiv r \vee (\sim p \vee q) \\
 &\equiv r \vee (p \rightarrow q) \\
 &\equiv r \vee (\sim q \rightarrow \sim p)
 \end{aligned}$$

- [15] 2. { **Medium-Hard** } Let  $\mathcal{U}$  be a set, and  $\mathcal{P}(\mathcal{U})$  denote its power set. For each subset  $A$  of  $\mathcal{U}$ , we can define a function  $f_A : \mathcal{P}(\mathcal{U}) \rightarrow \mathcal{P}(\mathcal{U})$  by

$$f_A(S) = A - S$$

Prove that  $f_A$  is one-to-one if and only if  $A = \mathcal{U}$ .

**Solution:** First we prove that if  $A = \mathcal{U}$  then  $f_A$  is one to one. Indeed, suppose that  $A = \mathcal{U}$ . If we have two subsets  $S, S'$  of  $\mathcal{U}$  such that  $S \neq S'$ , then there is an element  $x$  of  $\mathcal{U}$  that belongs to one of the subsets (say  $S$  without loss of generality) and not the other. Since  $x \in S$  and  $x \in A$ , this means that  $x \notin f_A(S)$ . Moreover, since  $x \notin S'$  and  $x \in A$ , it follows that  $x \in f_A(S')$ . Therefore  $f_A(S) \neq f_A(S')$ .

We now prove that if  $A \neq \mathcal{U}$  then  $f_A$  is not one to one. Suppose that  $A \neq \mathcal{U}$ . Then there is an element  $x$  of  $\mathcal{U}$  that does not belong to  $A$ . Let  $S = \emptyset$  and  $S' = \{x\}$ . Then  $f_A(S) = A - S = A$ , and  $f_A(S') = A - S' = A$  (since the only element of  $S'$  does not belong to  $A$ ). Therefore  $f_A$  is not one to one.

- [10] 3. { **Medium-Easy** } Prove that for any positive integer  $n$

$$\sum_{i=0}^n \frac{1}{i!} \leq 3 - \frac{1}{n}$$

Recall that  $0! = 1$ , and for  $n > 0$ ,  $n! = 1 * 2 * 3 * \dots * n$ . If you want, you can use the fact that for every  $n \geq 1$

$$\frac{1}{(n+1)!} \leq \frac{1}{n(n+1)}$$

without proving it.

**Solution:** We prove the fact by induction on  $n$ . The base case is the case  $n = 1$ . We have  $\sum_{i=0}^1 1/i! = 1/0! + 1/1! = 2$ , and  $3 - 1/1 = 2$ . So  $\sum_{i=0}^1 1/i! \leq 3 - 1/1$ .

We now deal with the induction step. Consider an unspecified positive integer  $n$ . Suppose that  $\sum_{i=0}^n 1/i! \leq 3 - 1/n$ . Then

$$\begin{aligned} \sum_{i=0}^{n+1} \frac{1}{i!} &= \left( \sum_{i=0}^n \frac{1}{i!} \right) + \frac{1}{(n+1)!} \\ &\leq 3 - \frac{1}{n} + \frac{1}{(n+1)!} \\ &\leq 3 - \frac{1}{n} + \frac{1}{n(n+1)} \\ &= 3 - \left( \frac{1}{n} - \frac{1}{n(n+1)} \right) \\ &= 3 - \left( \frac{n+1}{n(n+1)} - \frac{1}{n(n+1)} \right) \\ &= 3 - \frac{n}{n(n+1)} \\ &= 3 - \frac{1}{(n+1)} \end{aligned}$$

Hence the theorem holds by the principle of mathematical induction.

[10] 4. { **Medium** } Let  $p_0, p_1, p_2, p_3, \dots$  be an integer sequence defined by

$$\begin{aligned} p_0 &= 1, \quad p_1 = 2, \quad p_2 = 3, \quad \text{and} \\ p_n &= p_{n-1} + p_{n-2} + p_{n-3} \quad \text{for all } n \in \mathbf{Z}^+ \text{ where } n \geq 3. \end{aligned}$$

(For example,  $p_3 = 3 + 2 + 1 = 6$  and  $p_4 = 6 + 3 + 2 = 11$ .)

Prove that for all nonnegative integers  $n$ ,  $p_n \leq 3^n$ .

**Solution:** We use the strong form of mathematical induction. First the base cases:

- For  $n = 0$ ,  $p_0 = 1$  and  $3^0 = 1$ . Clearly  $1 \leq 1$ .
- For  $n = 1$ ,  $p_1 = 2$  and  $3^1 = 3$ . Clearly  $2 \leq 3$ .
- For  $n = 2$ ,  $p_2 = 3$  and  $3^2 = 9$ . Clearly  $3 \leq 9$ .

Now we consider the induction step. Pick an unspecified integer  $n \geq 3$ , and suppose that the theorem holds for  $0, 1, 2, \dots, n-1$ . Now,

$$\begin{aligned} p_n &= p_{n-1} + p_{n-2} + p_{n-3} \\ &\leq 3^{n-1} + 3^{n-2} + 3^{n-3} \\ &\leq 3^{n-1} + 3^{n-1} + 3^{n-1} \\ &= 3^n. \end{aligned}$$

Hence the theorem holds for every  $n \geq 0$  by the principle of mathematical induction.

[24] 5. { **Hard** } Figure 1 shows a sequential logic circuit (i.e. one with state held in D flip-flops). The circuit has inputs **up** and **down** and outputs  $q_0$  and  $q_1$ . We will refer to the outputs combined together as  $q$  and interpret  $q$  as a two's complement, binary value.

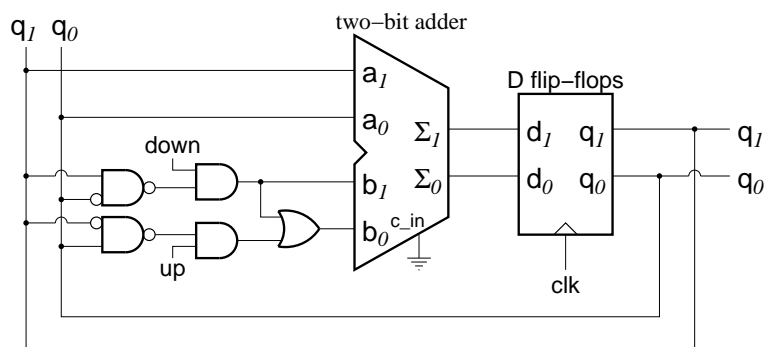


Figure 1: Sequential circuit for question 5

[6] a. Complete the table below as follows:

- For the column labeled  $q$ , write the value of  $q$  interpreted as a two's complement binary integer.
- Let us call  $b_1$  and  $b_0$  the two bits of the lower input to the adder in the circuit. For the columns labeled  $b_1$  and  $b_0$ , write the values of these signals as a boolean function of the inputs **up** and **down**, assuming that the values of  $q_0$  and  $q_1$  are as written on that row. You can abbreviate **up** as  $u$  and **down** as  $d$ .

We have filled in the first row as an example:

$q_1$	$q_0$	$q$	$b_1$	$b_0$
F	F	0	$d$	$u \vee d$
F	T			
T	F			
T	T			

**Solution :**

$q_1$	$q_0$	$q$	$b_1$	$b_0$
F	F	0	d	$u \vee d$
F	T	1	d	d
T	F	-2	F	u
T	T	-1	d	$u \vee d$

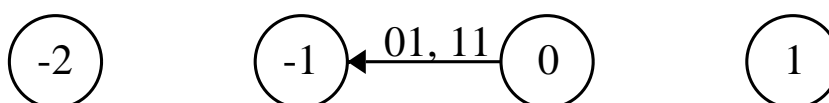
- [6] b. For each possible pairs of values of **up** and **down**, this circuit can be seen as computing a function of the input  $q$ . Fill in the following table, indicating the function that is being computed.

up	down	$f(q)$
F	F	$f(q) = q$
F	T	
T	F	
T	T	

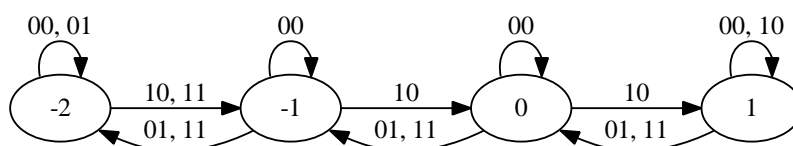
**Solution :**

up	down	$f(q)$
F	F	$f(q) = q$
F	T	$f(q) = q-1$ except $f(-2) = -2$
T	F	$f(q) = q+1$ except $f(1) = 1$
T	T	$f(q) = q-1$ except $f(-2) = -1$

- [9] c. Draw the state transition diagram (finite state machine) that corresponds to this circuit. Each state has been labeled with the corresponding value of  $q$  (interpreted as a two's complement signed integer). You need to add the arrows between states. Each arrow should be labeled with a pair of letters that causes this state transition to take place. For instance, we have labeled the one arrow  $01, 11$  to indicate that if the current state is  $0$ , and at the next clock tick we have  $up = 0$  and  $down = 1$ , or  $up = 1$  and  $down = 1$ , then the next state will be  $-1$ .



**Solution :**



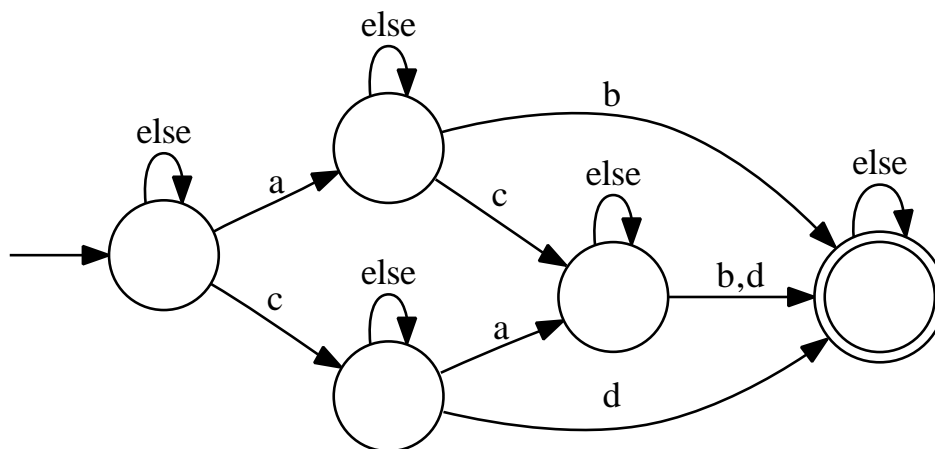
- [3] d. Give a **brief** explanation of what this circuit does. Your answer should be no longer than five simple sentences (shorter answers are possible).

**Solution :** This circuit either counts up ( $up = 1, down = 0$ ) or down ( $up = 0, down = 1$ ) without overflowing. That is, once its gets to the largest (smallest) possible value while going up (down), it remains at that value. If  $up = down = 0$  then nothing happens. The behaviour for the case where  $up = down = 1$  is just an accident.

- [8] 6. { **Medium** } Design a DFA that takes as input a string over the alphabet  $\Sigma = \{a, b, c, \dots, z\}$  and ends up in an accepting state if and only if the input string contains *either* the letter **a** followed some time later by the letter **b**, or the letter **c** followed some time later by the letter **d** (or both).

You may write the label **else** on an edge to indicate that the transition will be taken if the next character is any character other than one already used as a label on another edge that starts from the same state (i.e. the same way we did in class for the real numbers example).

**Solution :**



- [8] 7. { **Medium** } Prove or disprove the following statements about two functions  $f : B \rightarrow C$  and  $g : A \rightarrow B$ : if  $f$  is one-to-one and  $g$  is onto, then  $f \circ g$  is one-to-one.

**Solution :** This is false. Suppose that  $A = \{1, 2\}$ ,  $B = \{a\}$ , and  $C = \{\text{green}\}$ . Define  $g = \{(1, a), (2, a)\}$  and  $f = \{(a, \text{green})\}$ . Then  $f$  is one-to-one,  $g$  is onto, and yet  $(f \circ g)(1) = (f \circ g)(2)$ , which means that  $f \circ g$  is not one-to-one.

- [9] 8. { **Medium** } Write regular expressions that describe the following sets of strings:
- [3] a. All strings of letters whose first and last characters are both **a**, **or** whose first and last characters are both **o**.

**Solution :**  $\hat{a}[A-Za-z]^*a\$ | \hat{o}[A-Za-z]^*o\$$ .

- [6] b. All strings over the alphabet  $\{0, 1\}$  that do not contain the substring 011.

**Solution :** The trick to derive the regular expression is that once we have seen a 0, every other 1 afterwards must be followed by either at least one 0, or by nothing at all. Hence we get:  $\hat{1}^*(0+1?)*\$$  (this was tricky).