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1 Propositional Logic Tautologies, Contradictions, and Contingencies [4 marks]

For each propositional logic statement below, indicate (by circling the appropriate answer) whether it is a tautology, a contradiction, or a contingency (neither a tautology nor a contradiction).

These are worth [1 mark] each.

1. $p \rightarrow (q \rightarrow \sim p)$

Circle one: **tautology** **contradiction** **contingency**

2. $(\sim(p \oplus q)) \leftrightarrow (\sim q \oplus p)$

Circle one: **tautology** **contradiction** **contingency**

3. $((p \leftrightarrow q) \leftrightarrow r) \rightarrow (p \vee q \vee r)$

Circle one: **tautology** **contradiction** **contingency**

4. $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim s) \wedge \sim(p \vee \sim s)$

Circle one: **tautology** **contradiction** **contingency**

2 Circuit Properties [4 marks]

For each of the following statements, label them as **definitely true**, **possibly true**, or **definitely false** for any combinational circuit that can be modeled by one or more propositional logic statements. (That is, circle one of those options for each statement.)

Note: combinational circuits are the circuits we worked with in the first part of the course in which no path through the circuit loops back on itself.

These are worth [1 mark] each.

1. Every input to the circuit is connected to the input of a gate.

Circle one: **definitely true** **possibly true** **definitely false**

2. Some gate's output is connected to the output of another gate.

Circle one: **definitely true** **possibly true** **definitely false**

3. No input to the circuit is connected to the input of more than one gate.

Circle one: **definitely true** **possibly true** **definitely false**

4. The number of gates is less than or equal to the number of outputs of the circuit.

Circle one: **definitely true** **possibly true** **definitely false**

3 Representing Relationships with Predicate Logic [5 marks]

Let S be the set of all students and C be the set of all courses. Consider the following predicates:

- $Required(s, c)$ means course c is required for student s to graduate
- $Enrolled(s, c)$ means student s is enrolled in course c
- $Full(c)$ means course c has no room for further enrollment

1. Translate the following into predicate logic. [2 marks]

Any course that's full has at least one student enrolled in it.

2. Imagine you were using proof by contradiction to prove this theorem:

$\exists c \in C, \exists s \in S, Enrolled(s, c) \vee Full(c)$.

(a) State the assumption that would begin your proof (in predicate logic). Hint: any correct statement is acceptable; you need not simplify in any way. [1 mark]

(b) Given your assumption, what, if anything, would you know about any student and course, regardless of which particular student or course they are? (Write your answer in English, not predicate logic.) [2 marks]

4 Proof Critique [5 marks]

Let \mathbf{R} be the set of all real numbers, let \mathbf{Z} be the set of all integers and let \mathbf{Z}^+ be the set of all positive integers. The set of rational numbers, \mathbf{Q} , can be defined as the set of numbers $\frac{a}{b}$ for some $a, b \in \mathbf{Z}$ ($b \neq 0$). Without loss of generality, one can further stipulate that a and b have no factors in common. That is, there is no number $k \in \mathbf{Z}^+, k > 1$ that divides both a and b . A real number $x \in \mathbf{R}$ that is not rational is called irrational. (Note that any integer is a rational number since an integer $n = n/1$.)

1. Assuming its premises are true, can a proof that $\sqrt{4}$ is irrational be valid? Why or why not? [2 marks]

2. Indicate the step or steps in the proof below that are invalid or state here that the proof is valid. [3 marks]

Note that the proof makes use of the following lemma, which is valid.

Lemma: $\forall n \in \mathbf{Z}^+$, if n^2 is even then n is even.

Proof (by contradiction): $\sqrt{4}$ is irrational.

Assume $\sqrt{4}$ is rational (i.e., assume the negation of theorem).

Then $\sqrt{4} = \frac{a}{b}$ for some $a, b \in \mathbf{Z}$ ($b \neq 0$).

Further, assume that a and b share no common factors. Now,

$$\begin{aligned}(\sqrt{4})^2 &= \left(\frac{a}{b}\right)^2 \\4 &= \frac{a^2}{b^2} \\a^2 &= 4b^2\end{aligned}$$

Hence, a^2 is even. If a^2 is even then a is even (by given lemma) and $a = 2c$ for some $c \in \mathbf{Z}$. Substituting back into $a^2 = 4b^2$, we obtain

$$\begin{aligned}(2c)^2 &= 4b^2 \\4c^2 &= 4b^2 \\b^2 &= c^2\end{aligned}$$

Hence, b^2 also is even. Once again, if b^2 is even then b is even (by given lemma).

Thus, a and b share the common factor 2 which contradicts our assumption that they have no common factors.

Therefore, $\sqrt{4}$ is irrational

QED

5 Number Representation [9 marks]

Since you do not have a calculator, you may leave your answers as arithmetic expressions (e.g., $53 + 27 + 5$), but the correct answers should be obtainable from your expressions simply by entering them into a basic calculator.

For parts 1–3 below, consider the pattern of digits 11010001.

1. Could these digits be a signed binary integer? If so, assume they are and give their value in base 10. If not, explain why not. [2 marks]

Circle one: **YES** **NO**

If **YES**, give the base 10 value.

If **NO**, explain why not.

2. Could these digits be an unsigned binary integer? [1 mark]

Circle one: **YES** **NO**

3. Could these digits be an integer in base 10? [1 mark]

Circle one: **YES** **NO**

4. When using 8 bits, can we represent the most distinct values by interpreting the bits as an unsigned binary integer, as a signed binary integer, or as an unsigned binary fraction (with a $1/2$ s place, $1/4$ s place, ..., and $1/256$ ths place)? [3 marks]

Circle the representation that yields the most distinct values. If there is a tie, circle all tied representations:

Unsigned binary integer

Signed binary integer

Unsigned binary fraction

5. Briefly explain one compelling reason why representing binary numbers in base 16 would be more convenient for humans than base 256. [2 marks]

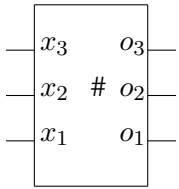
6 Circuit Design [17 marks]

Using the steps below, design a circuit that takes two 3-bit numbers as input and outputs the one with the most 1s in it. (That is, the three output bits are identical to and in the same order as the input with the most 1s. If both have the same number of 1s, the output may match either input.)

You may design your circuit using inverters, 2-input XOR and XNOR gates, 2- or 3-input AND, OR, NAND, and NOR gates, 2:1 multiplexers, and the modules described below.

1. Design a circuit that takes three inputs and produces three outputs, o_1 , o_2 , and o_3 . o_1 is 1 if and only if one or more of the inputs are 1s. o_2 is 1 if and only if two or more of the inputs are 1s. o_3 is 1 if and only if all three of the inputs are 1s. **[5 marks]**

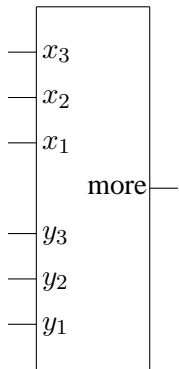
In the next part, use your circuit from part 1 as a module with the following symbol:



You should use this module in part 2, even if you do not complete part 1. Assume the module works as described in the problem statement for part 1.

2. Design a circuit that takes two 3-bit inputs and has a single output named *more*. The output is true if and only if the first input has at least as many 1s as the second. [7 marks]

In the next part, use your circuit from part 2 as a module with the following symbol:



You should use this module in part 3, even if you do not complete part 2. Assume the module works as described in the problem statement for part 2. Hint: this is actually the easiest part, particularly if you use multiplexers; you may want to complete it first!

3. Design the full circuit that takes two 3-bit inputs and outputs the one with the most 1s. **[5 marks]**

7 Proving Properties of Computing Systems [13 marks]

A decision algorithm is an algorithm that, when run on an input, answers either *yes* or *no*. Furthermore, there is a “correct” answer for each input, either *yes* or *no*, and the algorithm may or may not get the correct answer.

A decision algorithm is called sound exactly when: if the algorithm reports that the answer for an input is *yes*, the correct answer for that input is *yes*.

A decision algorithm is called complete exactly when: if the correct answer for an input is *yes*, then the algorithm reports that the answer for that input is *yes*.

(An algorithm that is not sound is “unsound”. An algorithm that is not complete is “incomplete”.)

Consider the following procedure to generate a new decision algorithm, which we call “zipping”: the algorithm resulting from zipping decision algorithm a_1 and decision algorithm a_2 is called zip_{a_1, a_2} and answers *yes* exactly when one or both of a_1 and a_2 answer *yes*.

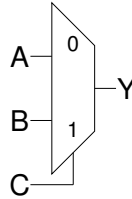
1. Prove *using a direct proof* that for any pair of algorithms a_1 and a_2 , if a_1 and a_2 are both sound, zip_{a_1, a_2} is sound. [4 marks]

2. Prove *using the contrapositive* that if zip_{a_1, a_2} is incomplete, then both a_1 and a_2 are incomplete. **[5 marks]**

3. Prove *using contradiction* that zipping an unsound algorithm with itself does not produce a sound algorithm. **[4 marks]**

8 Universality [10 marks]

A 2 input, 1 select line MUX is referred to as a 2:1 MUX. The task is to prove that a 2:1 MUX is universal. For a 2:1 MUX, the output Y is equal to input A when C is 0 and equal to input B when C is 1. The gate symbol is:



Note: For each of the next three parts of this problem, you may supply any otherwise unused MUX input with constant 0 or constant 1 (i.e., wire an input to ground or power).

1. Show how a 2:1 MUX can be used to implement an inverter. [2 marks]
2. Show how a 2:1 MUX can be used to implement a 2-input AND gate. [3 marks]
3. Show how a 2:1 MUX can be used to implement a 2-input OR gate. [3 marks]
4. Does what we requested in steps 1–3—showing implementations of each gate in terms of a multiplexer—prove that the 2:1 MUX is universal? Why or why not? [2 marks]

9 A Bit More Number Representation [4 marks]

Consider the 8-bit unsigned binary number 00011011.

1. Translate it to decimal. [2 marks]

2. Translate it to hexadecimal. [2 marks]

10 Abstract Proof [4 marks]

Describe in as much detail as possible what strategy you would use to prove the following theorem using a direct proof approach.

$$\forall y \in E, \exists z \in F, P(y, z) \leftrightarrow R(y, z).$$

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