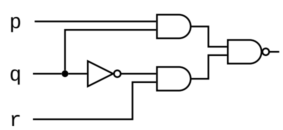
CPSC 121 Sample Midterm Examination
October 2007

[9] 1. Match each proposition in the left column with the logically equivalent proposition in the right column. Each proposition from the left column is equivalent to only one proposition in the right column. Since there are three propositions in the left column and five propositions in the right column, two of the propsitions in the right column will not be used when you make your matches. Justify your answers (i.e. show why the two propositions are logically equivalent).

a. $(p \vee q) \to r$

b. $(p \oplus q) \vee (p \wedge q) \vee r$

c. 

1. T
2. $(p \vee q) \vee (\sim q \wedge r)$
3. $p \vee (q \vee r)$
4. $(\sim q \wedge \sim p) \vee r$
5. $p \oplus q \oplus r$

Solution :

- a matches with 4
- b matches with 3
- c matches with 1

Valid justifications are truth tables, or showing that the two propositions are logically equivalent using logical equivalences

- a with 4

$$\begin{aligned} (p \vee q) \to r &\equiv \sim(p \vee q) \vee r && \text{Definition of} \to \\ &\equiv (\sim q \wedge \sim p) \vee r && \text{De Morgan's law} \end{aligned}$$

- b with 3

$$\begin{aligned} (p \oplus q) \vee (p \wedge q) \vee r &\equiv (p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge q) \vee r && \text{Definition of } \oplus \\ &\equiv (p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q) \vee r && \text{Commutative Law} \\ &\equiv (p \wedge (\sim q \vee q)) \vee (\sim p \wedge q) \vee r && \text{Distributive Law} \\ &\equiv (p \wedge T) \vee (\sim p \wedge q) \vee r && \text{Negation Law} \\ &\equiv p \vee (\sim p \wedge q) \vee r && \text{Identity Law} \\ &\equiv ((p \vee \sim p) \wedge (p \vee q)) \vee r && \text{Distributive Law} \\ &\equiv (T \wedge (p \vee q)) \vee r && \text{Negation Law} \\ &\equiv (p \vee q) \vee r && \text{Identity Law} \\ &\equiv p \vee (q \vee r) && \text{Associative Law} \end{aligned}$$

This was also equivalent to 2.

- c with 1

c	$\equiv$	$\sim((p \wedge q) \wedge (\sim q \wedge r))$	Translation from circuit diagram
	$\equiv$	$\sim(p \wedge (q \wedge \sim q) \wedge r)$	Commutative Law
	$\equiv$	$\sim(p \wedge F \wedge r)$	Negation Law
	$\equiv$	$\sim(F \wedge r)$	Domination Law
	$\equiv$	$\sim(F)$	Domination Law
	$\equiv$	T	Definition of $\sim$

[10] 2. Rules of inference

[3] a. Prove that the following rule (simplification) is a valid rule of inference:

$p \wedge q$

$\therefore\ p$

Solution : Show that $(p \wedge q) \rightarrow p$ is a tautology.

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

[7] b. Prove or disprove that the following argument is valid. Please clearly state the rule of inference you are using for each step.

1. $m \rightarrow n$
2. $k \rightarrow m$
3. $r \rightarrow q$
4. $g \wedge h$
5. $\sim r \rightarrow m$
6. $h \rightarrow \sim(k \vee q)$

$\therefore\ n$

Solution :

7.	h	Simplification - 4
8.	$\sim(k \vee q)$	Modus ponens - 6, 7
9.	$\sim k \wedge \sim q$	De Morgan's - 8
10.	$\sim r$	Modus Tollens - 3, 9
11.	m	Modus Ponens - 5, 10
$\therefore$	n	Modus Ponens - 1, 11

[6] 3. Translate each of the following English propositions into predicate logic. Assume that C is the set of all countries, M is the set of all continents, $B(x, y)$ means x shares a border with y, $P(x, y)$ means country x is in continent y, and $H(x)$ means country x is in the Northern Hemisphere.

[3] a. No country in the Northern Hemisphere borders a country in the Southern Hemisphere.

Solution : $\forall x \in C, \forall y \in C, H(x) \wedge \sim H(y) \rightarrow \sim B(x, y)$.

[3] b. There are some countries in the Northern Hemisphere that share a border but aren't in the same continent.

Solution :
$\exists x \in C, \exists y \in D,\ H(x) \wedge H(y) \wedge B(x, y) \wedge (\forall z \in M, \sim(P(x, z) \vee \sim P(y, z))$.

[11] 4. Representing numbers.

[2] (a.) Give the binary (bit-string) representation of the following base 10 numbers when represented as four-bit, signed numbers in two's complement notation.

(i.) 5:

Solution : 0101

(ii.) -5:

Solution : 1011

[2] (b.) Here is a truth table (with numbers 1 and 0 instead of T and F) for a circuit that adds two two-bit, **unsigned** numbers (i.e., $a_1a_0 + b_1b_0 = u_1u_0$). (It is split in two parts to fit on the paper better.)

a_1	a_0	b_1	b_0	u_1	u_0	s_1	s_0
0	0	0	0	0	0		
0	0	0	1	0	1		
0	0	1	0	1	0		
0	0	1	1	1	1		
0	1	0	0	0	1		
0	1	0	1	1	0		
0	1	1	0	1	1		
0	1	1	1	0	0		

a_1	a_0	b_1	b_0	u_1	u_0	s_1	s_0
1	0	0	0	1	0		
1	0	0	1	1	1		
1	0	1	0	0	0		
1	0	1	1	0	1		
1	1	0	0	1	1		
1	1	0	1	0	0		
1	1	1	0	0	1		
1	1	1	1	1	0		

What changes are required to this circuit for it to add two **signed, two's complement** numbers (ie., s_1s_0)? To answer this question you can fill in the s_1 and s_0 columns in the table or provide a careful explanation of what values these column should have, whichever you find easier. In either case, explain your answer carefully.

Solution : No changes are necessary: addition works exactly the same way, whether the numbers are signed or unsigned.

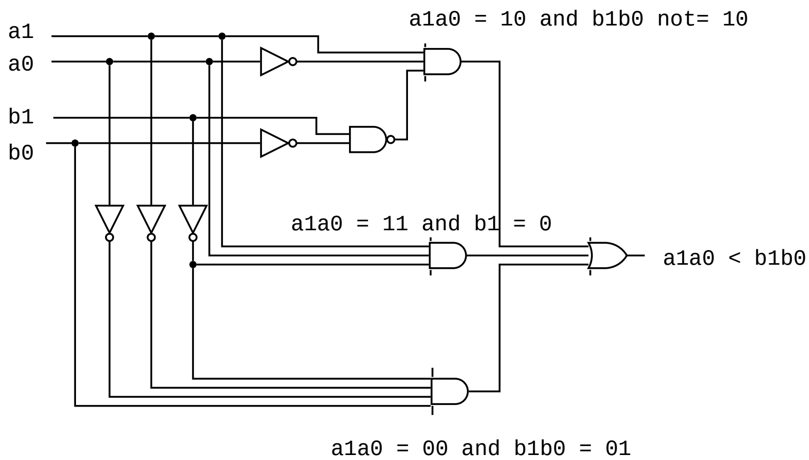
[7] (c.) Draw a circuit that takes two two-bit, signed, two's complement numbers a_1a_0 and b_1b_0 as input and computes an output of 1 if $a < b$ and 0 otherwise.

Solution : We want our circuit to output 1 in the following cases:

- $a_1a_0 = 10$ and $b_1b_0 \neq 10$.

- $a_1a_0 = 11$ and b_1b_0 is either 00 or 01.
- $a_1a_0 = 00$ and $b_1b_0 = 01$.

Thus the output will come from an OR gate with three inputs (one for each possibility).



[5] 5. Answer the following questions about these four sets.

$$\begin{aligned} A = &\ \{1, 2, ..., 10\} \\ B = &\ \{x \in A \mid x \text{ is odd}\} \\ C = &\ \{x \in A \mid x \text{ is divisible by 3}\} \\ D = &\ \{1, 2, 1, 3, 3, 3\} \end{aligned}$$

(a.) $A \cap B =$

Solution : $\{1, 3, 5, 7, 9\}$

(b.) $B \cup C =$

Solution : $\{1, 3, 5, 6, 7, 9\}$

(c.) $|B \cup D| =$

Solution : 6

(d.) $|D - B| =$

Solution : 1

(e.) $|\{T \mid T \subseteq C\}| =$

Solution : 8

[11] 6. Consider the following statement:

> An unsigned integer represented in binary is odd if and only if its rightmost (least-significant) bit is 1.

[3] (a) Rewrite the statement using predicate logic (quantifiers and predicates).

Solution: $\forall x \in \mathbf{N}, \sim \textit{even}(x) \leftrightarrow \sim \textit{lastbitis}(x, 1)$

[8] (b) Prove the truth of the statement using a direct proof. Part of the marks will be given for the structure of the proof; the rest will be given for the details.

Solution: Consider an unspecified non-negative integer x. The theorem is a biconditional, so we need to prove two implications.

- First we show that if x is odd, then its rightmost bit is 1. So assume that x is odd. Then there is some non-negative integer y such that $x = 2y + 1$. Now if $y = \sum_{i=0}^{n-1} b_i 2^i$, then $x = 1 + \sum_{i=1}^{n-1} b_{i-1} 2^i$. Hence the term in 2^0 in the binary representation of x is indeed 1.
- Now we prove that if the rightmost bit of x is 1, then x is odd. Indeed if $x = \sum_{i=0}^{n-1} b_i 2^i$ where $b_0 = 1$ then $x = 1 + 2\sum_{i=0}^{n-2} b_{i+1} 2^i$, and so x is odd.

[8] 7. There were 184 students registered in CPSC 121 in term 2 of last year. Using an indirect proof, prove that there were two CPSC 121 students who either had the same birthday, or birthdays that occur on 2 consecutive days.

Solution: We use a proof by contradiction. Suppose that no two CPSC 121 students either had the same birthday, or birthdays that occur on 2 consecutive days. Since a year has at most 366 days, this means that there were at most $366/2 = 183$ students. This contradicts the fact that there were 184 students in the course.