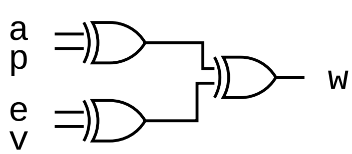
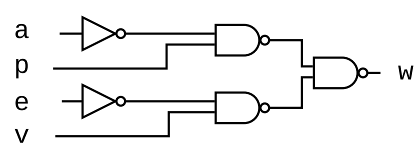
CPSC 121 Midterm
Thursday, October 25th, 2007

[12] 1. After the midterm, several friends are thinking about going to see a movie, either *Death at a Funeral* or *Rendition*. Let

a = Alice wants to go to the movie
e = Ebert wants to go to the movie
p = Peter wants to go to the movie
v = Volkov wants to go to the movie
w = Wolfgang wants to go to the movie
d = The movie is *Death at a Funeral*
r = The movie is *Rendition*

The table below contains various English sentences, predicates, and logic circuits. Each sentence, truth value of an expression, or output of a circuit describes a set of conditions where Wolfgang will go to a movie (that is, "w"). Identify those that are equivalent to each other. You will find 3 groups of two table entries that are equivalent, and two entries that are equivalent to no other entry. You should write your answer in the form "$A \equiv B$, $C \equiv D$, $E \equiv F$, G and H are unpaired."

Write your answers below (continue at the top of the next page). You should give a short justification (one or two sentences) for each pairing. **Do not spend more than 15 minutes on this question, unless you have already completed the rest of the examination.**

A.		B.	
C.	Wolfgang will go to a movie if and only if at least three of the other four people want to go to it.	D.	Wolfgang will go if and only if that makes the number of people in the group even.
E.	Wolfgang will only go to *Death at a Funeral* with Peter or to *Rendition* with Alice.	F.	$(d \wedge p) \vee (r \wedge a)$
G.	$(p \wedge \sim a) \vee (v \wedge \sim e)$	H.	$((a \vee e) \wedge (v \vee p)) \vee$ $((a \vee v) \wedge (e \vee p))$

Solution :

- E matches F: this is a direct English translation.
- B matches G: the circuit outputs the value $\sim(\sim(\sim a \wedge p) \wedge \sim(\sim e \wedge v))$. By DeMorgan's law and the double negation law, this is the same as $(\sim a \wedge p) \vee (\sim e \wedge v)$, which is the same as the proposition in G by the commutative law.
- A matches D: the only way for the output of the circuit to be true is if either 1 or 3 of the four inputs is/are true. Thus Wolfgang will go if either 1 or 3 other people go, i.e. if (and only if) this makes the total number of people even.
- C and H are unmatched.

[12] 2. Because he was busy grading a Computer Science midterm, Steve was unable to attend the department's Halloween party. He managed to obtain some information by talking to people who attended the party. Unfortunately, their memories were rather imprecise, and so all that Steve learned was the following:

- Either Ujjal or Wolfgang came dressed as a werewolf.
- Either Ronald or Stéphanie was dressed as a dragon.
- Priscilla and Quirinius both left early.
- If Ujjal came dressed as a werewolf, then Ronald came dressed as a dragon and Tom was dressed as a giant snake.
- If Ronald was dressed as a dragon, then Quirinius did not leave early.
- Tom was disguised as a giant snake if and only if neither Ronald nor Stéphanie was disguised as a dragon.

[4] (a) Rewrite each of these statements using propositional logic. Make sure to define the propositions you are using (that is, state something like: "u: Ujjal came dressed as a werewolf" before writing the proposition $\sim$u).

Solution : We will use the following propositions (I have ordered them alphabetically, but there was no need for you to do that):

p : Priscilla left early
q : Quirinius left early
r : Ronald was dressed as a dragon
s : Stéphanie was dressed as a dragon
t : Tom was dressed as a giant snake
u : Ujjal was dressed as a werewolf
w : Wolfgang was dressed as a werewolf

Given that, the statements can be rewritten as:

1. $u \vee w$
2. $r \vee s$
3. $p \wedge q$
4. $u \rightarrow (r \wedge t)$
5. $r \rightarrow \sim q$
6. $t \leftrightarrow \sim r \wedge \sim s$

[8] (b) Using your answer from part (a), known logical equivalences, and the rules of inference, prove that Wolfgang was dressed as a werewolf.

Solution : Here is one possible proof:

7.	q	simplification from 3
8.	$\sim\sim q$	by the double-negation law
9.	$\sim r$	modus tollens from 5 and 8
10.	$\sim(r \wedge t)$	using the definition of $\wedge$ from 9
11.	$\sim u$	modus tollens from 4 and 10
12.	w	disjunctive syllogism from 1 and 11

[6] 3. Let G be the set of all games of the Vancouver Canucks in the 2007/2008 Hockey season, and F be the set of all hockey fans. Furthermore, suppose we have a predicates *Saw(x,y)* that is true if fan x saw game y (in person, or on TV). You may assume that "first game" and "last game" are element of G that correspond to the first and last games of the Canucks' season respectively.

[3] (a) Translate into English the proposition

$$\exists g \in G\ \forall x \in F\ \mathit{Saw}(x, g) \vee \mathit{Saw}(x, \mathit{last\ game})$$

Solution : Here are a couple of possible answers:

> There is a specific game played by the Vancouver Canucks with the property that every hockey fan saw either that game or the Canuck's last game of the season, or both.

and

> There is a game played with the Vancouver Canucks with the property that everybody who did not see that game saw the Canuck's last game of the season.

[3] (b) Write the statement "Every fan who saw at least two different games saw the first game of the season" using predicate logic.

Solution : Here is one possible solution:

$$\forall x \in F \quad (\exists g1 \in G\ \exists g2 \in G\ \mathit{Saw}(x, g1) \wedge \mathit{Saw}(x, g2) \wedge g1 \neq g2) \rightarrow \mathit{Saw}(x, \mathit{first\ game})$$

Note that the parentheses on the left-hand side of the implication were required (removing them changes the meaning of the proposition).

[11] 4. Number representation

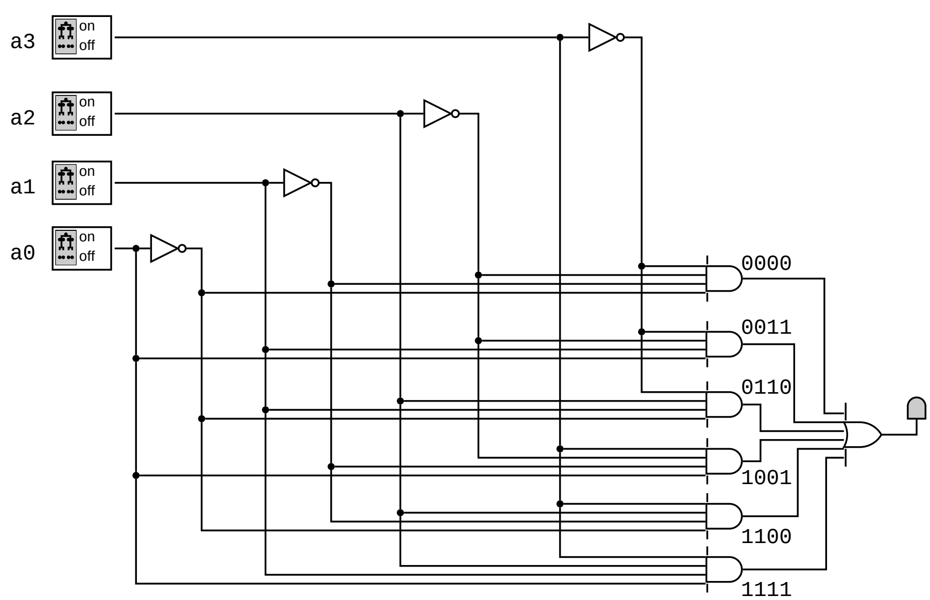
[3] a. Explain briefly why almost all modern computers represent integers using two's complement representation, instead of setting aside one bit to use as a sign (with 0 representing $+$, and 1 representing $-$).

Solution : The main reason is that using two's complement notation allows us to implement addition and subtraction of both unsigned and signed integers using one circuit. If we instead used a separate sign bit, then addition and subtraction (or even addition of numbers with different signs) would need to use completely different circuits.

[8] b. Design a circuit that takes as input a four-bit unsigned integer $a_3a_2a_1a_0$, and outputs 1 if that integer is divisible by 3. **Show your work**: we need to understand how you designed your circuit and why you believe it computes the correct answer. Hint: do **not** try to implement binary division. Instead, use the fact that there are only a few values for which your circuit should output 1.

A working circuit will be worth at least 6 marks out of 8. To get full marks, your circuit will also need to be reasonably elegant (a solution with only 7 gates exists).

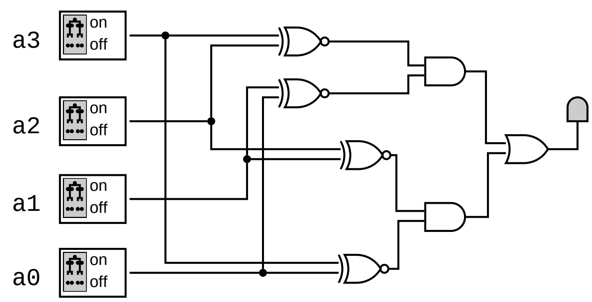
Solution : The only four-bit unsigned integers that are divisible by 3 are 0 (0000) **(Yes, 0 is divisible by 3!)**, 3 (0011), 6 (0110), 9 (1001), 12 (1100) and 15 (1111). One way to design the circuit would be to test for each possibility independently, and then combine them using an OR gate:



A more elegant solution would be to observe that the bit patterns listed above have at least one of the following two properties (and are the only bit patterns that satisfy one or both of them):

- The first two bits are the same, and so are the last two bits.
- The middle two bits are the same, and so are the "outside" two bits.

This leads us to the following more elegant solution:



[11] 5. Consider the following theorem:

> No matter what positive real number c we pick, there will be some positive integer n for which $3^n > c2^n$.

[3] a. Translate this theorem into predicate logic (that is, using predicates and quantifiers).

Solution : $\forall c \in \mathbf{R}^+ \, \exists n \in \mathbf{Z}^+ \; 3^n > c2^n$.

[8] b. Prove the theorem using a *direct proof*.

Solution : This theorem contains a universal quantifier, with an existential quantifier nested inside it. Hence, we start by considering an unspecified positive real number c. Now, we need to show how to choose a value of n for which $3^n > c2^n$.
Scratch work (not part of the answer as such, but you might have scribbled that on the back of the page): we want $3^n > c2^n$. That will be satisfied as long as $3^n/2^n > c$. That is, as long as $(3/2)^n > c$, which means $n > \log_{3/2} c$.
Back to the proof: pick any integer n larger than $\log_{3/2} c$ (for instance, $1 + \lceil \log_{3/2} c \rceil$ would do). Then $(3/2)^n > c$, which means that $3^n > c2^n$.

[8] 6. Using an *indirect proof*, prove that the equation $ax + b = c$ (where a, b, c are real numbers, and $a \neq 0$) has only a single solution for x.

Solution : We use a proof by contradiction. Suppose to the contrary that there are values of a, b and c, with $a \neq 0$, for which the equation $ax + b = c$ has at least two different solutions x_1 and x_2 for x. Then $ax_1 + b = c$ and $ax_2 + b = c$. This means that $ax_1 + b = ax_2 + b$, and hence that $ax_1 = ax_2$. Since $a \neq 0$, it therefore follows that $x_1 = x_2$. However this contradicts our assumption that $x_1 \neq x_2$. This means that the original theorem must be true.